ALGEBRAIC IDENTITIES IN THE THEORY OF NUMBERS

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1. Two identities. Elementary algebraic identities have frequently been used to establish arithmetical theorems whose proofs are not evident otherwise. Fermat, for example, noted that an odd integer is prime if and only if it is the difference of two squares in only one way, from which an interesting two-dimensional representation of the sieve of Eratosthenes is obtainable. Typical of another kind of result, Oltramare stated (1894) that every integer is a sum of five integer cubes, and this was seen to be an immediate consequence of the identity

$$6m = (m+1)^3 + 2(-m)^3 + (m-1)^3.$$

Similar cubic identities have been used in attempts to prove the conjecture that four cubes suffice.

Several such identities are special cases of formulas of the types

$$ncN = c_1 u_1^n + \cdots + c_{n+1} u_{n+1}^n, \tag{1}$$

$$ncN + c' = c_1 v_1^n + \cdots + c_n v_n^n, \tag{2}$$

in which n is any constant integer >1, N is an arbitrary integer, the u's, v's are variable integers (depending on N), and the constant coefficients $c, c', c_1, \cdots, c_{n+1}$ (independent of N) are polynomials, none identically zero, with integer coefficients, in $2n-2$ integer parameters. The degree in the parameters of c, c' is $n(n-1)$; that of each of the remaining constants, $n(n-2)$. For all N with the possible exception of certain divisors of c, the u's, v's are integers >0 if n is even, and may be chosen >0 if n is odd by changing the signs of some of the constants. In the exceptional case some of the u's, v's may be zero.

2. Explicit forms. To state explicit forms of (1), (2), we note that any integer N may be written $x^{n-1}y$, where x, $|x| \geqq 1$, is any $(n-1)$th-power divisor of N. Identically in x, y,

$$\sum_{r=1}^{n-1} (n, r) a_i^{n-r} b_i^r x^{n-r} y^r \equiv w_i,$$

$$(n, r) \equiv n!/r!(n-r)!, \qquad w_i \equiv (a_i x + b_i y)^n - a_i^n x^n - b_i^n y^n.$$

Provided the determinant of the system obtained from this identity by taking $i = 1, \cdots, n-1$ does not vanish, the system may be solved for the $x^{n-r}y^r$, $r = 1, \cdots, n-1$. The determinant is readily evaluated and the solution appears in a simple form. Write

$$A \equiv a_1 \cdots a_{n-1}, \qquad B \equiv b_1 \cdots b_{n-1},$$

$$d_{i,j} \equiv a_i b_j - a_j b_i, \qquad D \equiv \prod_{r=1, s>r}^{n-1} d_{r,s},$$

$$A_i \equiv A/a_i, \qquad B_i \equiv B/b_i, \qquad D_i \equiv D/D^{(i)},$$

where $D^{(i)}$ is the product of all $d_{r,s}$ in which one of r, s is i, and $a_1, \cdots, a_{n-1}$, $b_1, \cdots, b_{n-1}$ are such that $ABD \neq 0$. Then

$$nABDx^{n-1}y = \sum_{i=1}^{n-1} (-1)^{i-1} A_i B_i^2 D_i w_i,$$

for (1) with $N = x^{n-1}y$; and

$$nABDN + \sum_{i=1}^{n-1} (-1)^{i-1} a_i^n A_i B_i^2 D_i$$
$$= \sum_{i=1}^{n-1} (-1)^{i-1} A_i B_i^2 D_i (a_i + b_i N)^n - \left(\sum_{i=1}^{n-1} (-1)^{i-1} A_i B_i^2 D_i b_i^n \right) N^n$$

for (2), obtained from the preceding formula by $x = 1$, $y = N$.

For $n = 3$ the formula (2) is

$$a_1 a_2 d_{1,2}(3b_1 b_2 N + a_1 b_2 + a_2 b_1) = a_2 b_2^2 (a_1 + b_1 N)^3 - a_1 b_1^2 (a_2 + b_2 N)^3 + b_1^2 b_2^2 d_{1,2} N^3,$$

which gives Oltramare's identity when $a_1 = 2$, $a_2 = b_1 = b_2 = 1$, $N = m - 1$. The choice $a_1 = a_2 = b_1 = 1$, $b_2 = 2$, $N = m$ gives

$$6m + 3 = 4(m + 1)^3 - (2m + 1)^3 + 4m^3;$$

whence every integer is of the form

$$x^3 + y^3 + 4(z^3 + w^3)$$

with x, y, z, w integers. This follows by reducing $6m + 3 + t^3$, $t = 0, \cdots, 5$, modulo 6. Thus if $3 + t^3 = 6r + s$, $0 \leqq s < 6$, the identity gives

$$6(m + r) + s = 4(m + 1)^3 - (2m + 1)^3 + 4m^3 + t^3,$$

or, with m replaced by $m - r$,

$$6m + s = t^3 + (-2m + 2r - 1)^3 + 4(m - r + 1)^3 + 4(m - r)^3;$$

and since the residues s are a permutation of the integers t, the result follows. From the last with $s = 0, \cdots, 5$ it is seen that x, y, z, w may all be chosen different from zero with the following possible exceptions:

$$s = 0, \quad m = 4, 5; \qquad s = 1, \quad m = 10, 11;$$
$$s = 2, \quad m = 20, 21; \qquad s = 3, \quad m = 0, -1;$$
$$s = 4, \quad m = 0, -1; \qquad s = 5, \quad m = 0, 1.$$

For $n = 4$, $a_1 = a_2 = b_2 = b_3 = 1$, $a_3 = b_1 = 2$, $N = m - 1$ we have

$$24m - 3 = (2m - 1)^4 + 2(m - 1)^4 - 12m^4 - 6(m - 1)^4.$$

Whence, as in the preceding example, all integers of the forms $24m + 6$, 13, 21, 22 are represented in

$$x^4 + y^4 + 2z^4 - 6w^4 - 12u^4$$

in which the integers $x, \cdots, u$ may all be chosen greater than zero with the possible exceptions $24m+6$, $m=2, 3, 4$; $24m+13$, $m=-1, 0, 1$; $24m+21$, $m=-2, -1, 0$; $24m+22$, $m=-2, -1, 0$.

As an example for $n=5$, we take $a_1=a_2=a_3=a_4=-b_1=b_4=1$, $b_2=-b_3=2$.

$$60y = 8(y-1)^5 + 8(y+1)^5 - (2y-1)^5 - (2y+1)^5 + 2(2y)^5;$$

whence it follows (by referring to a table of fifth powers for residues of x^5+2w^5 mod 60) that every integer is of the form

$$x^5 + y^5 + z^5 + 2w^5 + 2u^5 + 8v^5 + 8t^5,$$

with $x, \cdots, t$ integers.

3. Derived identities. Several further identities may be obtained from (1) by differentiations and use of the relations $a_iA_i=A$, $b_iB_i=B$. For example,

$$ADx^{n-1} = \sum_{i=1}^{n-1} (-1)^{i-1} A_iB_iD_i[(a_ix + b_iy)^{n-1} - b_i^{n-1}y^{n-1}];$$

$$0 = \sum_{i=1}^{n-1} (-1)^{i-1} A_iD_i[(a_ix + b_iy)^{n-2} - b_i^{n-2}y^{n-2}];$$

$$Dx^{n-r} = \sum_{i=1}^{n-1} (-1)^{i-1} a_i^{r-2} B_iD_i(a_ix + b_iy)^{n-r}, \quad 1 < r \leqq n;$$

$$0 = \sum_{i=1}^{n-1} (-1)^{i-1} D_i a_i^{n-t-3} b_i^t, \quad 0 \leqq t < n-3.$$

Identities similar to (1) may be constructed from the expansion of

$$\left(\sum_{i=1}^{n} a_ix_i\right)^m - \sum_{i=1}^{n} a_i^m x_i^m.$$

Examples of the use of differentiation are given incidentally in the next section.

4. Rational identities. Writing y for N in the second identity, we set the left member equal to the arbitrary t,

$$nABDy + A\sum_{i=1}^{n-1} (-1)^{i-1} a_i^{n-1} B_i^2 D_i = t,$$

solve for y, and substitute into the original identity. Then

$$t = \sum_{i=1}^{n-1} (-1)^{i-1} A_iB_i^2D_ix_i^n - B\left(\sum_{j=1}^{n-1} (-1)^{j-1} A_jB_jD_jb_j^{n-1}\right)x^n, \tag{3}$$

in which

$$nABDx_i = b_i t + AB\left(nDa_i - \sum_{j=1}^{n-1} (-1)^{j-1} a_j^{n-1} B_j D_j\right),$$

$$nABDx_n = t - A \sum_{j=1}^{n-1} (-1)^{j-1} a_j^{n-1} B_j^2 D_j,$$

with $ABD \neq 0$.

For $b_j = 1, j = 1, \cdots, n-1$, the identity (3) takes a remarkably simple form. After some straightforward reduction, we find

$$t = \sum_{i=1}^{n-1} (-1)^{i-1} A_i D_i' y_i^n - (-1)^n D' y_n^n, \tag{4}$$

in which

$$nAD'y_i = t + AD'[na_i - (a_1 + \cdots + a_{n-1})],$$
$$nAD'y_n = t - AD'(a_1 + \cdots + a_{n-1}),$$
$$D' \equiv \prod_{r<s} (a_r - a_s), \qquad A \equiv a_1 \cdots a_{n-1},$$

with $AD' \neq 0$, and D_i' is obtained from D' by deleting all $a_r - a_s$ in which one of r, s is i.

From (4) it follows that every rational number t is of the form

$$\sum_{i=1}^{n} c_i y_i^n, \tag{5}$$

in which $c_i = (-1)^{i-1} A_i D_i'$, $c_n = (-1)^{n-1} D'$, $AD' \neq 0$, $(i \neq n)$. If $a_1, \cdots, a_{n-1}$ are rational integers, the c_j are rational integers all different from zero, and the y_i are n rational numbers, all different. (If $y_i = y_j$, $i \neq j$, then $AD' = 0$; a contradiction.) In numerical examples, any common factor k of the coefficients c_i in (5) may be suppressed, with a corresponding change in the values of the y_i. For if t be replaced by kt, we may divide out k in (4) and in the expressions for the y_i, y_n. This device is used in obtaining the identities (6), (7).

If t in (4) is (temporarily) a continuous real variable, we may differentiate successively with respect to t. Replacing t by $AD't$ in the results, we get

$$\sum_{i=1}^{n-1} (-1)^{i-1} A_i D_i y^s + (-1)^{n-1} D' y_n^s = 0, \qquad n > 1, \tag{6}$$
$$s = 0, 1, \cdots, n-2,$$

in which

$$y_i = t + na_i - (a_1 + \cdots + a_{n-1}),$$
$$y_n = t - (a_1 + \cdots + a_{n-1}).$$

Here we have dropped the denominator n appearing in the expressions for

y_i, y_n on dividing throughout by AD', as is permissible since the $n-1$ equations (6) are homogeneous in the y's. The constants and variables in (6) are unrestricted.

For the special case of (6) in which all the letters denote rational integers, let g be the G.C.D. of

$$a_1 + \cdots + a_{n-1}, \qquad na_i - (a_1 + \cdots + a_{n-1}), \qquad i = 1, \cdots, n-1,$$

and h the G.C.D. of D', A_iD_i', $i=1, \cdots, n-1$; and define the g_i, h_i by

$$a_1 + \cdots + a_{n-1} = -gg_n,\ na_i - (a_1 + \cdots + a_{n-1}) = gg_i,\ (-1)^{i-1}A_iD_i' = hh_i,$$
$$(-1)^{n-1}D' = hh_n.$$

Then, replacing t by gt in (6) and dropping the denominator g, we get

$$\sum_{i=0}^{n} h_i(t + g_i)^s = 0, \qquad s = 0, 1, \cdots, n-2. \tag{7}$$

In this the h_i are non-zero rational integers and the g_i are n different rational integers.

We give some examples of the preceding formulas. From (7) with $n=6$, $a_1=1$, $a_2=2$, $a_3=3$, $a_4=-1$, $a_5=-2$ we get

$$(t+5)^p + 5(t-3)^p + 10(t+1)^p = (t-5)^p + 5(t-3)^p + 10(t-1)^p,$$
$$p = 0, 1, 2, 3, 4.$$

The choice $n=5$, $a_i=i$ gives

$$4(t+1)^p + 4(t-1)^p = 6t^p + (t+2)^p + (t-2)^p, \qquad p = 0, 1, 2, 3.$$

For $n=4$ the identity (4) may be written

$$t = bc(b-c)x^4 + ca(c-a)y^4 + ab(a-b)z^4 + dw^4,$$

in which $d \equiv (a-b)(b-c)(c-a)$, $abcd \equiv k \neq 0$,

$$-4kx = t - k(3a - b - c), \qquad -4ky = t - k(-a + 3b - c),$$
$$-4kz = t - k(-a - b + 3c), \qquad -4kw = t + k(a + b + c).$$

By an obvious transformation this is equivalent to

$$t = fx^4 + gy^4 + hz^4 - (f + g + h)w^4:$$
$$d \equiv r^3fgh(1+f)(1-g)(1+fg) \not\equiv 0;$$
$$-4dx = t - rd(1 - f - 3g - fg),$$
$$-4dy = t - rd(1 + 3f + g - fg),$$
$$-4dz = t - rd(1 - f + g + 3fg),$$
$$-4dw = t + rd(3 + f - g + fg),$$

with the condition $fgh+f+g+h=0$. These are of interest in connection with

the statement (1832) of Libri* that each of the forms

$$2x^4-20y^4+30z^4-12w^4, \qquad x^4-y^4+3z^4-3w^4$$

represents all rational numbers with x, y, z, w rational numbers. The first of these is given by the choice $a=5, b=2, c=1$; the second by $f=1, g=-1, h=3$. As in the derivation of (7), it follows from the first that all rational numbers are also represented in $x^4-10y^4+15z^4-6w^4$ with rational x, y, z, w.

The formulas have numerous applications to diophantine analysis. To illustrate one of them, we transform the identity (4) for $n=3$, which may be written

$$\begin{aligned} t &= bx^3-ay^3+(a-b)z^3, \qquad ab(a-b)\neq 0: \\ 3ab(a-b)x &= t+ab(a-b)(2a-b), \\ 3ab(a-b)y &= t+ab(a-b)(-a+2b), \\ 3ab(a-b)y &= t-ab(a-b)(a+b). \end{aligned}$$

If the identity is multiplied throughout by $[3ab(a-b)]^3$, and in the result, after expansion of the left side, a, b, t are replaced by $a^{3n}, b^{3n}, 3^{3n-3}t^{3n}$ we get the indicated three-parameter solution of

$$x_1^3+x_2^3+x_3^{3n}+3x_4^{3n}=y_1^3+y_2^3+y_3^{3n}+3y_4^{3n}:$$

$$\begin{aligned} x_1 &= b^{3n}[3^{3n-3}t^{3n}+a^{3n}b^{3n}(a^{3n}-b^{3n})(2a^{3n}-b^{3n})], & x_3 &= 3a^3b^6t, \\ x_2 &= a^{3n}[3^{3n-3}t^{3n}-a^{3n}b^{3n}(a^{3n}-b^{3n})(a^{3n}+b^{3n})], & x_4 &= 3a^5b^4t, \\ y_1 &= a^{3n}[3^{3n-3}t^{3n}+a^{3n}b^{3n}(a^{3n}-b^{3n})(-a^{3n}+2b^{3n})], & y_3 &= 3a^6b^3t, \\ y_2 &= b^{3n}[3^{3n-3}t^{3n}-a^{3n}b^{3n}(a^{3n}-b^{3n})(a^{3n}+b^{3n})], & y_4 &= 3a^4b^5t. \end{aligned}$$

Thus for n any integer >0, the equation has a three-parameter solution in integers $x_1, \cdots, y_4$. If in the result of differentiating the equation with respect to t we replace n by $2n$, and in the resulting equation t, a, b by $a^{6n-1}, b^{6n-1}, 3^{6n}t^{6n-1}$, and make the same successive substitutions in the values of $x_1, \cdots, y_4$, we get a three-parameter integer solution of

$$x_1^2+x_2^2+x_3^{6n-1}+3x_4^{6n-1}=y_1^2+y_2^2+y_3^{6n-1}+3y_4^{6n-1}.$$

If $x_1', \cdots, y_4'$ are the values of $x_1, \cdots, y_4$ in the case $n=1$ of the former equation, derivation with respect to $t(n=1)$ gives

* See L. E. Dickson, History of the Theory of Numbers, vol. 2, 1920, p. 727, where only one of Libri's forms is cited. Both are given in Journal für Mathematik (Crelle's Journal), vol. 9, 1832, p. 292, with two evident misprints in the exponents. Libri's method in this paper is Lagrange's, based on the product-theorem for norms of algebraic numbers. He does not give the values of the variables; these may be found by substituting in the formulas above. Libri notes that since each form represents all rational numbers, the product of two forms of either kind is a form of the same kind; and he infers that the forms repeat under multiplication, as do Lagrange's norms. But the variables in Lagrange's forms are independent; in all forms such as Libri's and those of the present paper they are polynomials in a parameter, and hence are not independent. The forms therefore do not repeat under multiplication as the term is now commonly understood.

$$bX_1^2 + aX_2^2 + aX_3^2 + 3aX_4^2 = aY_1^2 + bY_2^2 + bY_3^2 + 3bY_4^2:$$

$$X_1 = tx_1', \quad X_2 = tx_2', \quad X_3 = ab^2x_3', \quad X_4 = a^2b^2x_4',$$

$$Y_1 = ty_1', \quad Y_2 = ty_2', \quad Y_3 = a^3by_3', \quad Y_4 = a^2b^2y_4'.$$

As a last application, we may consider the constants in the identities as numbers in any algebraic number field, and the variables as numbers in the same or another field. Expression of the numbers concerned in the canonical (basis) form then gives, on equating coefficients of linearly independent numbers, a system of equations (or identities) equivalent to the original, the constants and variables in the system being rational numbers. A parametric solution of the system, in rational numbers, is given by equating coefficients in the solution of the original equation. Without this reduction we see for example that any number of an algebraic number field A of degree m is of the form (5), with the c_i rational integers and the y_i numbers in A.

PARTITION OF SPACE

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1. The problem. It is well known that n straight lines divide the plane into at most $(n^2+n+2)/2$ regions. This often appears in a more familiar guise as the answer to the question: *how many pieces can be obtained from a round flat cheese by exactly n straight cuts*? We will generalize this to both Euclidean r-space —an r dimensional cheese—and projective r-space, obtaining in each case the number of p dimensional regions into which the space is partitioned, for $p=0, 1, \cdots, r$.

2. A geometrical lemma. Consider the configuration formed by n hyperplanes in Euclidean r-space (E_r) having general intersection.* Let $F_r(n)$ be the number of r dimensional regions in the configuration. We will show that

$$F_r(n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r}.$$

Let us examine our configuration more closely; it consists of a finite number of 0-cells (points), 1-cells (edges), $\cdots$, r-cells (polyhedroids). Let $M_r(p, n)$ be the number of p-cells in the configuration; thus, $M_r(r, n) = F_r(n)$. We define $F_0(n)$ to be 1 for all n.

LEMMA.

$$M_r(p, n) = \binom{n}{r-p} F_p(n + p - r).$$

* By this we mean that every set of k hyperplanes ($r-1$ dimensional subspaces) intersect in an $r-k$ dimensional plane; if $r-k$ is negative, the intersection is void. In the plane, general intersection means that no two lines are parallel and no three are concurrent.

Consider an arbitrary p-plane (p dimensional plane) of the configuration. It is formed by the intersection of $r-p$ of the n hyperplanes; this leaves $n-(r-p)$ hyperplanes that cut the p-plane, forming in turn $n+p-r$ $(p-1)$-planes. These too have general intersection in the p-plane—which is itself an E_p—and therefore divide it into $F_p(n+p-r)$ p-cells. Every set of $r-p$ hyperplanes intersect in a p-plane, so that there are precisely $\binom{n}{r-p}$ p-planes in the configuration; this gives the value of $M_r(p, n)$ as above.

3. An application of combinatorial topology. We now make use of a result from elementary topology. If a configuration drawn on an r-sphere* consists of α_0 points, α_1 edges, $\cdots$, then, independent of the configuration drawn,

$$\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \cdots = \begin{cases} 2 \text{ if } r \text{ is even} \\ 0 \text{ if } r \text{ is odd.} \end{cases}$$

The value of this sum is known as the Euler Characteristic; if $r=2$, this reduces to the Euler Polyhedral Formula.

A corresponding formula holds for unbounded Euclidean r-space, namely:

$$\sum_{k=0}^{r} (-1)^k \alpha_k = (-1)^r$$

To prove this, add the point at infinity to E_r—producing a configuration for which $\alpha_0=1$, $\alpha_k=0$ if $k>0$—and map E_r on to the r-sphere with the point at infinity going into the north pole. Since the Euler Characteristic of the r-sphere is $1+(-1)^r$, we remove our added point, leaving $(-1)^r$ as the Euler number of E_r.

To apply this to our present problem, set $\alpha_p = M_r(p, n)$, whence

$$\sum_{p=0}^{r} (-1)^p M_r(p, n) = (-1)^r,$$

or, by the lemma,

$$\sum_{p=0}^{r} (-1)^p \binom{n}{r-p} F_p(n+p-r) = (-1)^r,$$

and replacing $r-p$ by k,

$$\sum_{k=0}^{r} (-1)^k \binom{n}{k} F_{r-k}(n-k) = 1. \tag{1}$$

THEOREM 1.

$$F_r(n) = \sum_{\lambda=0}^{r} \binom{n}{\lambda} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r}.$$

* An r-sphere is the r dimensional surface given by the equation

$$x_0^2 + x_1^2 + x_2^2 + \cdots + x_r^2 = 1.$$

Proof. Since (1) is a recurrence relation with initial conditions $F_0(n)=1$, there is a unique solution. We need only show that the stated form of $F_r(n)$ satisfies (1) for all r and n.

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}\left[\sum_{\lambda=0}^{r-k}\binom{n-k}{\lambda}\right]=\sum_{k=0}^{r}\sum_{\lambda=0}^{r-k}(-1)^k\binom{n}{k}\binom{n-k}{\lambda}$$

$$=\sum_{k=0}^{r}\sum_{\lambda=0}^{r-k}(-1)^k\binom{n}{k+\lambda}\binom{k+\lambda}{k}$$

$$=\sum_{j=0}^{r}\sum_{k=0}^{j}(-1)^k\binom{n}{j}\binom{j}{k}$$

$$=\sum_{j=0}^{r}\binom{n}{j}\delta_0^j=\binom{n}{0}=1.$$

COROLLARY 1. If $r\geqq n$, $F_r(n)=2^n$.

This means that if the dimension of the embedding space is high enough, every additional hyperplane merely halves each of the regions previously formed.

COROLLARY 2.

$$M_r(p,n)=\sum_{k=r-p}^{r}C_k\binom{n}{k}\quad\text{where}\quad C_k=\binom{k}{r-p}.$$

For $r=1, 2, 3, 4$, we have:*

$$F_1(n)=1+n$$

$$F_2(n)=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}=(n^2+n+2)/2$$

$$F_3(n)=(n^3+5n+6)/6$$

$$F_4(n)=(n^4-2n^3+11n^2+14n+24)/24.$$

Thus we see that a large spherical Edam cheese can be divided into exactly $(n^3+5n+6)/6$ pieces by n straight cuts.†

4. Projective r-space. The same technique can be applied to the problem of determining the number of r-cells obtained by slicing up *projective* r-space by n hyperplanes. The only difference encountered is that we must now use the Euler number of projective r-space, $[1+(-1)^r]/2$—*i.e.*, 1 if r is even, 0 if r is odd. If we denote by $M_r'(p,n)$ the number of p-cells obtained, and by $F_r'(n)$ the number of r-cells, the lemma still holds and we have:

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* It might be observed that an arbitrary polynomial $f(x)$ can be uniquely expressed as $f(x)\equiv\sum a_k\binom{x}{k}$; this seems to be the natural form for the functions with which we are here concerned.

† See problem E554, this MONTHLY, vol. 50, p. 59.

THEOREM 2.

$$F_r'(n) = \sum_{\lambda=0}^{r} \left\{ \frac{1 + (-1)^{r+\lambda}}{2} \right\} \binom{n}{\lambda} = \sum_{\lambda=0}^{[r/2]} \binom{n}{r - 2\lambda}.$$

COROLLARY.

$$M_r'(p, n) = \sum_{k=r-p}^{r} C_k \left\{ \frac{1 + (-1)^{r+k}}{2} \right\} \binom{n}{k} = \sum_{\lambda=0}^{[r-p/2]} C_{r-2\lambda} \binom{n}{r - 2\lambda},$$

where C_k is defined as in corollary 2, namely $C_k = \binom{k}{r-p}$.

For example*

$$F_1'(n) = n, \qquad F_2'(n) = \binom{n}{2} + \binom{n}{0} = (n^2 - n + 2)/2.$$

5. Bounded regions of E_r. Returning again to the configuration in E_r, we obtain the number of *bounded* r-cells, $F_r^*(n)$, by a repetition of the process this time using the Euler number 1.

THEOREM 3.

$$F_r^*(n) = \sum_{\lambda=0}^{r} (-1)^{r+\lambda} \binom{n}{\lambda} = \binom{n-1}{r}.$$

COROLLARY.

$$M_r^*(p, n) = \sum_{k=r-p}^{r} (-1)^{r+k} C_k \binom{n}{k} = \frac{r+1}{n+p-r} \binom{r}{p} \binom{n}{r+1}.$$

The simplifications are obtained by using the identity

$$\binom{n}{r+1} - \binom{n-1}{r} = \binom{n-1}{r+1}.$$

In conclusion, we observe that from any two of the theorems, the third may be inferred. For we have the relation

$$F_r'(n) = \frac{F_r^*(n) + F_r(n)}{2} = F_r^*(n) + \frac{F_r(n) - F_r^*(n)}{2}.$$

Geometrically, this means that in passing from Euclidean space to projective space, we count only half of the unbounded regions as being distinct.

* See W. B. Carver, The polygonal regions into which a plane is divided by n straight lines, this MONTHLY, vol. 48, p. 668.

THE RATE OF INTEREST IN INSTALMENT CONTRACTS

C. H. FISCHER, University of Michigan

1. Introduction. In recent years there has been a growing tendency to devote more attention in courses in the mathematics of finance to the subject of instalment buying and, in particular, to the determination of the rate of interest involved in an instalment contract. Instalment contracts generally contain no mention of a rate of interest—instead, it is stated that a certain flat sum called a *carrying charge* is to be added to the principal and this sum divided by the number of payments yields the payment size. The purpose, then, in computing an index such as the rate of interest must be merely to shed more light upon the transaction, perhaps to point out how intrinsically costly it is to buy "on time." We shall try to show here that the interest rate as generally computed is not the proper one to measure the real burden of the debt to the debtor but that it, instead, usually overstates it. A more suitable index for this purpose is then developed.

2. Typical solution. To clarify the issue and for the sake of concreteness. let us consider a typical instalment contract. Suppose that a debt of \$100 is to be discharged by means of twelve monthly payments, the size of each being determined by adding a carrying charge of \$8 to the original debt and dividing the sum by 12. The usual method of finding the interest rate is to equate the principal of \$100 to the present value, at the unknown rate of interest, of the twelve payments of \$9 each. The resulting equation, solved by interpolation in an interest table, yields an interest rate of 15.45%, effective.

3. Defect in the usual method. The above method of solution involves the implicit assumption that "money is worth" the unknown interest rate, that is, that all sums of money of any size can be immediately invested for any length of time at this uniform rate. This assumption is closely realized in fact as far as the finance company is concerned, and hence this is a very reasonable determination of the rate of interest earned by a company engaged in the issuance of such loans. From the point of view of the debtor, however, this assumption does not fit the facts. The ordinary individual does not have an opportunity to invest money at such a high rate of interest—in fact, he may find it difficult to earn any interest on small sums invested for short periods. Since in this application the basic assumption is invalid, the computed interest rate which rests upon it cannot be a proper measure to use in the case of the debtor.

To get a better idea of the true burden placed upon the debtor by his instalment contract, the possible alternative investments of his payments should be considered. Thus, if he had the opportunity to invest each \$9 payment at, say, 2% simple interest up to the termination date of the contract, he would be able to accumulate a total of \$108.99. This is the ultimate sum which he foregoes in making his monthly payments directly to the finance company, and this sum

suggests itself as the measure for which we are seeking. Since in this illustration the contract period is exactly one year and the principal is \$100, it is reasonable to state that the interest rate paid by the debtor is 8.99%; that is, he sacrifices the equivalent of \$108.99 one year hence in payment for a loan of \$100 contracted at the present time. Similarly, had the debtor been able to invest his payments at 4% instead of 2%, he would have been able to accumulate \$109.98 at the end of the year and it might well be argued that the true interest rate he paid on this \$100 loan was 9.98%.

An individual might find it almost impossible to obtain any interest on such small sums invested for such short periods. In this case it would be immaterial to the debtor whether he were required to pay over each \$9 to the finance company or were permitted to keep it on hand to the end of the year and pay all twelve instalments at that time. In any event his payments would amount to only \$108.00 at the end of the year and it would be fair to state that he paid only 8% interest.

4. New definition of the rate of interest. Let us, then, define *the rate of interest paid by the debtor as that rate of interest at which the original principal would accumulate, over the contract period, to an amount equal to the sum of the instalment payments, each accumulated to the termination date of the contract at the highest investment rate available to the debtor.*

This definition permits the use of either simple or compound interest for either or both of the interest rates involved. In practice, the type of interest used for the debtor's investment rate would be that which is actually available to him, while the choice of the type of interest paid by the debtor could be made arbitrarily.

5. Examples of resulting formulas. Three examples of formulas which would follow from the above definition are given below.

A. Both rates at compound interest. Let us define the symbols:

r = nominal interest rate, compounded m times per year, available to the debtor as an investor;
n = length of the contract in years;
k = carrying charge per unit of original principal;
m = number of instalment payments per year;
i = effective rate of interest paid by the debtor.

It is then easy to show that i is determined by the following equation:

$$(1 + i)^n = \frac{1 + k}{mn} s_{\overline{mn}|r/m}.$$

B. Debtor's investment rate at simple interest. If, in the above definitions of symbols, the meaning of r be altered so that

r = simple interest rate per year available to the debtor as an investor,

we find that i is determined by the equation

$$(1 + i)^n = (1 + k) \cdot [1 + r(mn - 1)/2m].$$

C. Both rates at simple interest. If we now alter the definition of i so that i = simple interest rate per year paid by the debtor, i is determined by

$$1 + ni = (1 + k) \cdot [1 + r(mn - 1)/2m].$$

In this latter case we note that there is a linear relationship between i and r. In general, the interest rate paid by the debtor as defined above is a monotonic increasing function of the effective rate available to the debtor in the role of investor.

6. Numerical illustration. The following table shows how in two numerical examples the interest rate paid by the debtor varies with the rate available to him as an investor. In both cases it is assumed that the debt will be discharged by means of twelve monthly payments. In the first example the carrying charge is \$8 per \$100 of principal; in the second example it is \$20 per \$100.

Debtor's investment rate compounded semi-annually	Effective rate paid by debtor	
	Case I	Case II
	Carrying charge \$8	Carrying charge \$20
0%	8.00%	20.00%
1	8.50	20.55
2	8.99	21.10
3	9.49	21.65
5	10.48	22.76
10	12.98	25.54
15	15.50	28.34

By the usual methods, assuming that the debtor's investment possibilities were equal to those of the finance company, his effective interest rate would be 15.45% for Case I and 41.30% for Case II.

ITERATION OF THE ϕ FUNCTION

W. H. MILLS, Aberdeen Proving Ground

1. Introduction. In the January 1943 number of this MONTHLY Dr. Harold Shapiro divided all positive integers into classes by means of the iteration of the Euler ϕ function as follows: Let

$$\phi^0(x) = x,$$
$$\phi^n(x) = \phi[\phi^{n-1}(x)], \tag{1}$$

where $\phi(x)$ represents the number of integers less than x and relatively prime to x. If $x=\Pi_i p_i^{\alpha_i}$, where the p_i are distinct prime factors of x, then

$$\phi(x) = \prod_i p_i^{\alpha_i-1}(p_i - 1). \tag{2}$$

From (2) it follows that $\phi(x)$ is even and less than x if $x>2$. Therefore if $x\geqq 2$ there exists a number n such that $\phi^n(x)=2$. Then x is said to be of class n.

Dr. Shapiro observed that the smallest number in every class up to the eighth was a prime and raised the question whether the smallest number in each class was a prime. It will be shown that this is not the case.

2. The smallest number. The simplest case in which the smallest number of a class is composite is the class 32.

THEOREM: $(2^{16}+1)^2$ *is the smallest number in class* 32.*

Proof: Let $N=(2^{16}+1)^2$. From (2) and the fact that $2^{16}+1$ is a prime it follows that

$$\phi(N) = 2^{16}(2^{16}+1),$$
$$\phi^k(N) = 2^{33-k} \quad \text{for} \quad 2 \leqq k \leqq 32,$$
$$\phi^{32}(N) = 2.$$

Therefore N is of class 32.

Let P be a number of class 32 smaller than N. Let $\phi^m(P)=2^{\alpha_m}A_m=2^{\alpha_m}\Pi_j p_{jm}^{\beta_{jm}}$ where A_m is an odd integer, and the p_{jm} are the distinct prime factors of A_m. Let $r_m=A_m/\phi(A_m)$ if $m\geqq 1$ and $r_0=P/\phi(P)$. $r_m\geqq 1$ for all values of m.

If $1\leqq m\leqq 31$, $\alpha_m\geqq 1$ since $\phi(x)$ is even for $x>2$, and

$$\phi^{m+1}(P) = \phi[\phi^m(P)] = \phi(2^{\alpha_m}A_m) = 2^{\alpha_m-1}\phi(A_m),$$
$$\frac{\phi^m(P)}{\phi^{m+1}(P)} = \frac{2A_m}{\phi(A_m)} = 2r_m.$$

Furthermore

$$\frac{\phi^0(P)}{\phi^1(P)} = \frac{P}{\phi(P)} = r_0.$$

Now

$$2^{32} + 2^{17} = N - 1 \geqq P = \phi^{32}(P)\prod_{i=0}^{31}\frac{\phi^i(P)}{\phi^{i+1}(P)} = 2^{32}\prod_{i=0}^{31} r_i.$$

Therefore

* The smallest numbers in classes 11, 12, and 24 are $(2^4+1)\cdot 137$, $(2^4+1)(2^8+1)$, and $(2^8+1)(2^{16}+1)$ respectively. Furthermore the numbers $(2^{16}+1)^k$ where $1\leqq k\leqq 15$ are the smallest numbers in their respective classes.

$$1 + 2^{-15} \geqq \prod_{i=0}^{31} r_i, \tag{3}$$

and

$$1 + 2^{-15} \geqq r_m \quad \text{if} \quad 0 \leqq m \leqq 31. \tag{4}$$

From (2) it follows that

$$r_m = \frac{A_m}{\phi(A_m)} = \prod_j \frac{p_{jm}}{p_{jm} - 1} \geqq \frac{p_{jm}}{p_{jm} - 1}.$$

Therefore

$$1 + 2^{-15} \geqq \frac{p_{jm}}{p_{jm} - 1},$$

$$p_{jm} \geqq 2^{15} + 1. \tag{5}$$

Let a be the largest number for which $\phi^a(P)$ is not a power of 2. $0<a<32$ since $\phi^{32}(P)=2$ by our assumption that P is of class 32, and $\phi^0(P)=P$ cannot be a power of 2 since the only power of 2 in class 32 is 2^{33} which is greater than N. Then

$$\phi^{a+1}(P) = \phi[\phi^a(P)] = 2^{\alpha_a - 1} \prod_j p_{ja}^{\beta_{ja}-1}(p_{ja} - 1)$$

which must be a power of 2. Therefore $\beta_{ja}=1$ for all values of j, and all the p_{ja} are of the form 2^n+1. But, since $2^{32}+1$ is known to be composite, the only prime of this form less than N that satisfies (5) is the Fermat prime $2^{16}+1$. Therefore $A_a=2^{16}+1$.

The only number of the form $2^s(2^{16}+1)$ in class 32 is $2^{17}(2^{16}+1)>N$, and therefore $a\neq 0$.

Now $\phi^a(P)=\phi[\phi^{a-1}(P)]=2^{\alpha_{a-1}-1}\Pi_j p_{ja-1}^{\beta_{ja-1}-1}(p_{ja-1}-1)=2^{\alpha_a}(2^{16}+1)$. It is impossible for $\phi^a(P)$ to contain $(2^{16}+1)^2=N$ as a factor since in that case $P\geqq N$. Therefore one of the p_{ja-1} is a prime of the form $2^s(2^{16}+1)+1$ where $1\leqq s\leqq 16$. Let $R_s=2^s(2^{16}+1)+1$. Then

R_s is divisible by 3 if $s\equiv 0 \pmod 2$,
R_s is divisible by 5 if $s\equiv 1 \pmod 4$,
R_s is divisible by 17 if $s\equiv 3 \pmod 8$,
R_s is divisible by 257 if $s\equiv 7 \pmod{16}$,

and R_{15} is divisible by 23. Therefore R_s is not a prime for $1\leqq s\leqq 16$. It follows that N is the smallest number in class 32.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON DETERMINANTS AND HADAMARD'S INEQUALITY

RICHARD BELLMAN, University of Wisconsin

It is easy to verify directly the familiar theorem that the product of two numbers, each the sum of two squares, can be represented as the sum of two squares, namely,

$$(1) \qquad (a_1^2 + a_2^2)(b_1^2 + b_2^2) = (a_1b_2 - a_2b_1)^2 + (a_1b_1 + a_2b_2)^2.$$

This has an interesting interpretation. Let (a_1, a_2), (b_1, b_2) be two points A, B, and O the origin. Then $(-a_2, a_1)$ is a point C, and OC is perpendicular to OA. The area of triangle OAB is $\frac{1}{2}OA \cdot OB \sin\theta$, where θ is angle (OA, OB), or $\frac{1}{2}\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)} \sin\theta$; the area of triangle OBC is $\frac{1}{2}OB \cdot OC \cos\theta$ or $\frac{1}{2}\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)} \cos\theta$. Adding the squares of their areas and expressing these areas by determinants we have

$$(2) \qquad (a_1^2 + a_2^2)(b_1^2 + b_2^2) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & b_2 \\ -a_2 & a_1 \end{vmatrix}^2.$$

We thus obtain equality (1) and in addition we have the case $n=2$ of Hadamard's inequality

$$D^2 \leqq \prod_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}^2 \right),$$

where D is the determinant $|a_{ij}|$, $i, j = 1, \cdots, n$.

We wish to show that this process can be extended to higher spaces and similar equalities can be obtained, giving as a by-product Hadamard's inequality for the general case. For simplicity we shall restrict ourselves to the case $n=3$.

Consider a tetrahedron $OA_1B_1C_1$, O the origin. Denote the coordinates of A_1, B_1, C_1 by (a_1^1, a_2^1, a_3^1), (b_1^1, b_2^1, b_3^1), and (c_1^1, c_2^1, c_3^1), respectively, and the lengths OA_1, OB_1, OC_1 by a, b, c, respectively. The volume V_1 of the tetrahedron is $\frac{1}{6}h_1ab \sin\theta_1$, where h_1 is the altitude from C_1 to the plane OA_1B_1 and θ_1 is the angle (OA_1, OB_1) in the triangle OA_1B_1. In determinant form the volume

$$V_1 = \pm \frac{1}{6} \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^1 & c_2^1 & c_3^1 \end{vmatrix}.$$

Now consider two other points $C_2(c_1^2, c_2^2, c_3^2)$ and $C_3(c_1^3, c_2^3, c_3^3)$ such that $OC_1 = OC_2 = OC_3$ and OC_1, OC_2, OC_3 are mutually perpendicular. Let V_2 be the volume of the tetrahedron $OA_1B_1C_2$, V_3 the volume of $OA_1B_1C_3$, h_2 the altitude from C_2, and h_3 the altitude from C_3. It is easy to show that

$$h_1^2 + h_2^2 + h_3^2 = OC_1^2 = c_1^{1^2} + c_2^{1^2} + c_3^{1^2} = c^2,$$

and thus

$$V_1^2 + V_2^2 + V_3^2 = \frac{1}{36}(h_1^2 + h_2^2 + h_3^2)a^2b^2 \sin^2\theta_1 = \frac{1}{36}a^2b^2c^2 \sin^2\theta_1.$$

Also

$$V_1^2 + V_2^2 + V_3^2 = \frac{1}{36}\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^1 & c_2^1 & c_3^1 \end{vmatrix}^2 + \frac{1}{36}\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix}^2 + \frac{1}{36}\begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^3 & c_2^3 & c_3^3 \end{vmatrix}^2.$$

This is sufficient to show Hadamard's inequality for three dimensions, but we are interested in an equality of the form (2).

Now for any position C_1, C_2, C_3 of C, let B vary so that $OB_1 = OB_2 = OB_3$ and OB_1, OB_2, OB_3 are mutually perpendicular, and denote the coordinates of B_2 and B_3 by (b_1^2, b_2^2, b_3^2) and (b_1^3, b_2^3, b_3^3), respectively.

Considering the nine possible volumes resulting from both B and C varying in the prescribed manner, with A_1 fixed, we see that

$$\sum_1^9 V_i^2 = \frac{1}{36}a^2b^2c^2(\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3),$$

where θ_k is the angle (OA_1, OB_k), $k = 1, 2, 3$. But the lines OB_1, OB_2, OB_3 constitute a set of axes and the θ_k are direction angles; thus

$$\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 2.$$

Therefore we obtain

$$\sum_{i,j}{}' \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^i & b_2^i & b_3^i \\ c_1^j & c_2^j & c_3^j \end{vmatrix}^2 = 2a^2b^2c^2.$$

We can remove the asymmetry with respect to A_1 by forming the corresponding identities with B and C fixed and then adding the three resulting equations, obtaining

$$\sum_{i,j,k}{}' \begin{vmatrix} a_1^i & a_2^i & a_3^i \\ b_1^j & b_2^j & b_3^j \\ c_1^k & c_2^k & c_3^k \end{vmatrix}^2 = 6a^2b^2c^2.$$

This process can clearly be continued to higher dimensions.

THE MORLEY TRIANGLE: A NEW GEOMETRIC PROOF

H. D. GROSSMAN, New York City

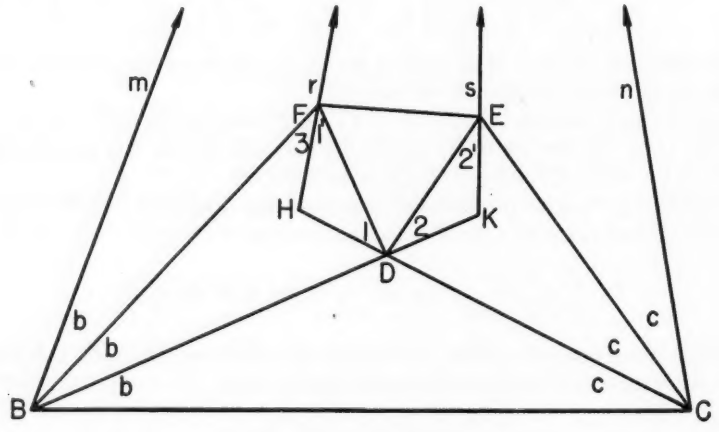
THEOREM. *If the angles of a triangle be trisected, the intersections of the pair of trisectors adjacent to each side determine an equilateral triangle.*

Proof. Let the triangle have base BC and angles $3a$, $3b$, $3c$. Let BDK, BF, CDH, CE be angle-trisectors. E is determined by making angle $CDE = 60° + b$ and F by making angle $BDF = 60° + c$. Then

$$\text{angle } EDF = 360° - (180° - b - c) - (60° + b) - (60° + c) = 60°.$$

Also

$$\text{angle } BFD = 180° - (60° + b + c) = 60° + a = \text{angle } CED.$$



Since D is equidistant from BF and CE, $DF = DE$ and triangle DEF is equilateral. Angle $1 = (60° + c) - (b + c) = 60° - b$. Similarly angle $2 = 60° - c$. Through F draw line r making angle $1' =$ angle 1. Through E draw line s making angle $2' =$ angle 2. Angle $3 = (60° + a) - (60° - b) = a + b$. Angle $mr = (a + b) - b = a$. Similarly angle $sn = (a + c) - c = a$. Angle $mn = 180° - 3b - 3c = 3a$.

It remains only to prove that the lines m, n, r, and s converge to a point. The line KF joins the vertices of two isosceles triangles and therefore bisects angle K. Then in triangle $mBKs$ the bisector of angle ms passes through F and being parallel to r, coincides with it. Similarly in triangle $rHCn$ the bisector of angle rn passes through E and being parallel to s, coincides with it.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1942-43

Kappa Mu Epsilon, Louisiana State University

We have been rather active this semester. The following topics were presented at our meetings:

Games of Chance, by K. L. Nielsen.

A proof that "conic sections" are conic sections, by F. A. Rickey.

Beta and Gamma functions, by Sam Cunningham.

The pendulum problem, by Carson Jeffries.

The brachistochrone problem in the calculus of variations, by Joseph Pryor.

Sam Cunningham won the Senior Award this year, and Robert Anding won the Freshman Honors Award. Sixty-four members have been initiated this year. Officers for next year are: President, Leland Morgan; Vice-President, Nina Nichols; Secretary, Gloria McCarthy; Treasurer, Julia Weil; Historian, William Wray; Faculty Adviser, Marelena White; Corresponding Secretary, Houston T. Karnes.

Kappa Mu Epsilon, Central Michigan College

The *Pythagorean Club* of Central Michigan College of Education became the *Michigan Beta Chapter of Kappa Mu Epsilon* on April 25, 1942. Regular business meetings were held once a month. The topic at one of the meetings was *Mathematics and the Navy*, by Mr. L. H. Serier. Several other meetings were devoted to the solving of mathematical puzzles. Two social gatherings were held in the College Den, at which time new members were initiated. The annual picnic was held June 5 at the city park. Officers for 1942–43 and 1943–44 were elected as follows: President, Jennie Master (42–43), Paul Brown (43–44); Vice-President, Paul Brown (42–43), Zelda Montague (43–44); Recording Secretary, Eleanore Mucynski (42–43), Robert Mark (43–44); Corresponding Secretary, Nicoline Bye (42–44); Treasurer, Jack Fiebing (42–43), Kenneth Miller (43–44). Dr. C. C. Richtmeyer acted as Faculty Adviser.

Pi Mu Epsilon, Lehigh University

Program meetings were held once a month throughout the year, and one open lecture by a speaker from outside the University was sponsored. This lecture was entitled *The motion of the spinning projectile*, by E. J. McShane, from the Aberdeen Proving Grounds and the University of Virginia.

There was a final initiation banquet at which the newly elected president,

Mr. Wright, presided. Officers for the past year and officers elect are as follows: President, Maynard Arsove (42–43), Robert Wright, Jr. (43–44); Secretary, Stanley Caplan (42–43), Ralph Evans (43–44); Treasurer, C. S. Bennett (42–43), Robert W. Logan (43–44); Faculty Adviser, Tomlinson Fort.

Mathematics Club, Wellesley College

The Mathematics Club of Wellesley College held three meetings during the year. At the first meeting, held in October, talks were given by those students who had held mathematical jobs during the summer. The subjects of these talks were as follows:

The application of mathematics to astronomy, by Jean North.

A draftswoman in a locomotive designing plant, by Elizabeth Bird.

A mathematical job at the Quonset Naval Base, by Elizabeth Weibel.

Testing bullets dimensionally, by Elizabeth Ann Wilson.

The topics presented at the November and April meetings were:

Probability and its applications, by Professor E. B. Mode of Boston University.

The algebra of logic, by Professor G. D. Birkhoff of Harvard, who showed how propositions of logic could be turned into algebraic expressions which are maneuverable according to certain rules.

A dinner was given before Professor Mode's talk. The third meeting, held at Wellesley in April, was a joint meeting with other clubs of the Greater Boston Intercollegiate Mathematics Clubs Association. The colleges represented were Harvard University, Regis College, Boston University, and Boston College. The officers of the club for 1942–43 were: President, June Nesbitt; Vice-President, Elizabeth Ann Wilson; Treasurer, Martha Adams; Junior Executive, Elizabeth Bird; Sophomore Executive, Ann Pettingell; Secretary, Phyllis Fox; Faculty Adviser, Professor L. P. Copeland.

Mathematics Club, Kansas State College

The Mathematics Club of Kansas State College is organized each year by *Pi Mu Epsilon*. After that the Club carries on with the advice of a faculty member who acts as program chairman. Meetings are held once a month. During the past year, students in the Civilian Pilot Training Program gave several talks on air navigation. Other meetings were taken up with talks by undergraduates in mathematics and engineering. Most of the talks came under one of the two general headings:

Applications of calculus, analytic geometry, and higher algebra to problems in engineering.

Solution of problems posted on the mathematics bulletin board.

The Club's officers this year were: President, Virginia Bell; Vice-President, Jean Burnette; Secretary, Aileen Hostinsky; Treasurer, Ann Dueser; Program Chairman, Dr. D. T. Sigley. (Report submitted by H. C. Fryer, Director, *Pi Mu Epsilon*.)

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Concise Spherical Trigonometry with Application and Reviews of Solid Geometry and Plane Trigonometry. By J. R. Hammond. Boston, Houghton Mifflin Co., 1943. 13+256 pages. $2.00.

The present crisis has produced a large number of texts on spherical trigonometry, ranging from brief syllabi to extensive treatises. The volume under review is original in many respects. The aim of the author is two fold: to acquaint the reader with a dependable guide, with a minimum of trustworthy preparation. The subject is introduced through Solid Geometry, no previous knowledge of which is presupposed. Theorems involving incidence properties are given, followed by considerable detail on spherical triangles and polar relations on the sphere. Knowledge of logarithms and of elementary plane trigonometry are presupposed.

The entire discussion of spherical trigonometry is based on one method, that is, of dividing every triangle into one or more right triangles. Napier's Rules are treated in great detail; only these elementary formulas are employed in each case.

A second feature is a prescribed scheme of arrangement of the details of computation, to be followed in detail in each case. The text proper is later amplified by a readable explanation of the means employed to provide the data for each problem, including a generous fund of celestial and terrestrial applications, mostly utilitarian, based on the author's extensive experience with technical problems. Some of these demand considerable general preparedness, a fairly high grade of wholesome interested maturity. Two chapters are added to provide more discussion of the ambiguous cases, first from a geometric point of view then by a more analytic treatment of the formulas, now derived. This part follows more closely the traditional treatment, including the Law of Sines, Law of Cosines, and their various corollaries. This Alternative Method is the only case in which any formula more complicated than those included in Napier's Rules is employed.

An appendix describes the sextant, the compass, the chronometer, transit, *etc.*, and a fairly full discussion of various kinds of time.

The style of the book is uniformly consistent; an effort is made to acquaint the reader with the elements of spherical trigonometry. It demands a receptive attitude on the part of the reader, and assumes a reasonable effort is being made to achieve that result.

VIRGIL SNYDER

Elementary Mathematics for the Machine Trades. By J. J. Weir. New York and London, McGraw-Hill Book Co., 1943. 8+193 pages. $1.60.

This is a text-book covering simple arithmetic, trigonometry and miscellaneous subjects useful to machine-shop workers. A machinist, however, would find most of the material too elementary, but the thousands of unskilled men and women who have been drawn into the war plants should find help in the simplified presentation of everyday problems provided they have an instructor's assistance. For the man who needs the simple arithmetic of the first part of the book, the range is too great. Probably with the aid of a good teacher, the pupil may be drawn from the fog of illiteracy of arithmetic, through decimals, simple algebra, geometry, pulleys, belts, verniers and gear trains to the use of trigonometric tables and functions. The treatment of the micrometer, height gage, vernier, caliper and graduated dials is very well done and amply illustrated. The books may be particularly useful to those who have covered all these subjects some time in the past, and now, when the prospect of practical use for bits of information presents itself, have motivation to relearn.

SAMUEL LERNER

College Mathematics. By W. L. Hart, W. A. Wilson and J. I. Tracey. Boston, Heath, 1943. 7+875 pp. $4.00.

This text, designed to meet the curricular requirements of the college programs of the army and navy below the level of calculus, is in three parts: 1. algebra; 2. trigonometry; 3. analytic geometry. Part 1 is a selection made by W. L. Hart from his *College Algebra, Revised Edition* giving a review of elementary algebra and the material in advanced algebra most useful in calculus. Part 2 is a selection with slight revision from *Trigonometry, Solid Geometry and Spherical Trigonometry* by W. W. Hart and W. L. Hart which omits solid geometry but includes tables. Part 3 is a reprinting of W. A. Wilson and J. I. Tracey: *Analytic Geometry; Alternate Edition.*

The requirements of the armed forces are here efficiently met in a single volume. Because of the great number of topics which must be touched, especially in the navy V-12 courses, a further selection and compression will probably have to be made by those who use this book in that program. It occurs to the reviewer that in spherical trigonometry, which is required by the navy but for which very little time is allowed in the standard curriculum, it might have been wiser to treat the oblique triangle by breaking it up into two right triangles (except when three sides or angles are given where, if the case is treated at all, the law of cosines may be used). Modern navigation requires exclusively the solution of the case: given two sides and the included angle. Several of the most popular methods in actual use solve it by means of two right triangles (*Hydrographic Office Publications* 208, by Dreisenstock, and 211, by Ageton; Weems, *Line of Position Book*; Hughes, *Sea and Air Navigation* (British)).

W. W. FLEXNER

Students' Handbook of Elementary Physics. By R. B. Lindsay. New York, The Dryden Press, 1943. 15+382 pages. $2.25.

R. B. Lindsay's "Handbook of Elementary Physics" is of unique design. It consists of three main parts, first a brief text of general elementary physics (about 100 pages), then a dictionary of physical terms and finally a collection of useful physical formulas, physical constants and mathematical tables. The text part is unusual in that it contains no mathematical formulas whatsoever. These are all relegated to the last two sections. The author feels that the student will get a better initial grasp of the principles and laws of physics if he is encouraged to state them in everyday English, rather than the abbreviated symbolism of mathematics. The idea of requiring the student to know how to state the laws of physics in his native tongue is fundamentally sound. However, the important thing is for him to be able to give a clear and accurate statement of the conditions under which the laws are valid and the operational meaning of the terms involved rather than just a long hand quotation of the mathematical formulas. This is not accomplished by simply omitting the symbolic mathematical formulas.

The style of writing in the text is clear and concise and will probably appeal to the student. Inasmuch as the handbook will usually be used to supplement a regular textbook, its brevity is not an objection. An interesting feature of the text is the frequent use of simple experiments which the student can perform himself to illustrate the points being discussed.

The dictionary part of the handbook contains the definitions of about 1000 physical terms together with appropriate diagrams and defining relations. The definitions might not all receive a perfect mark if found on an examination paper, but they do provide a useful source of first aid to the student who has difficulty in extracting bits of information from long winded texts. The table of physical formulas will be very popular with the students, but its pedagogical value is doubtful. The author warns that the formulas are not to be used without being completely understood. Yet in many cases it is difficult to see how a student can have a complete grasp of the meaning of the particular formula without being able to write it down instantly or derive it for himself in a few seconds.

The mathematical background needed by the student in using the handbook is negligible. To understand the text he need only know what "proportional to" and "average" means. If he is ambitious enough to attempt the problems he should know a little algebra. He is encouraged by the author to use the powers of 10 method for obtaining rough numerical answers. The formulas in the last two sections of the handbook contain trigonometric and exponential functions only occasionally, and integrals and derivatives only in one or two instances where they cannot be avoided.

H. HURWITZ, JR.

Learning to Navigate. By P. W. H. Weems and W. C. Eberle. The Pitman Publishing Co., 1943. 135 pp., 73 Figs. $2.00.

This is a second edition enlarged by more than sixty pages and forty illustrations.

The war has, of course, created an urgent demand for thousands of sea and air nagivators, many of whom will have little time for preparation. This book is intended to provide a foundation of the essentials of navigation for both sea and air.

After a bit of history of an early voyage, an account of the development of the compass and some elementary definitions, there follow a discussion of charts and how to use them and chapters on the magnetic and gyro compass, piloting and dead reckoning, celestial navigation, time, almanacs, sextants, line of position, log-book and air navigation.

This last is a new chapter. It discusses the various types of aeronautical charts, contact flying, dead reckoning, the use of mechanical computers, the graphical solution of wind-drift problems and celestial air navigation.

Radio navigation and meteorology are considered beyond the scope of the book.

Throughout the text only the shortest and simplest processes are explained. Dead reckoning is carried out directly on the chart instead of by logarithms or traverse tables. The line of position is derived by H.O. 214 or Weems' *Line of Position Book*. The latter uses cosecant and secants, but, to avoid frightening the beginner with trigonometrical terms, these are called simply *A* and *B*.

For the quickest determination of latitude and longitude in the air the use of Weems' *Star Altitude Curves* is recommended. For all other methods either the Nautical Almanac or Air Almanac is required. The authors advise the use of the Air Almanac for both sea and air.

In the process of condensation from larger works to the limited scope of this book there have resulted some inaccuracies of statement and inadequacies of explanation. However, these are not serious and the beginner may be assured that all the fundamental principles are sound.

FREDERICK SLOCUM

1000 Pre-flight Problems. By W. H. Thompson and M. L. Aiken. New York, Harper and Brothers, 1943. 15+160 pages. $0.88, paper; $1.20, cloth.

This pamphlet provides systematic drill in the solution of numerical problems connected with fundamental concepts in physics and mathematics that are directly or indirectly used in aviation. The scheme is to give a statement explaining briefly what the problem is, then follows the formula concerned, with the unknown appearing explicitly. Functions may contain radicals and simple trigonometric ratios. The derivation of the formulas are not given, nor are any answers to the problems. Readers are encouraged to use logarithms and the slide rule, but neither is required.

VIRGIL SNYDER

Solution of the Trisection Problem. By E. Vennigerholz. Moscow, Idaho, E. Vennigerholz, 1943. 16 pages. $1.25.

This little pamphlet offers not only what is claimed to be a solution of the problem of trisection, but also of various related problems. After a few pages of remarks, two constructions for the trisectors of a given angle are presented.

(a) Given two lines intersecting at O, at any given angle. With O as center and any given radius k describe a circle, meeting the sides of the given angle in AA' and BB', and the internal bisector of the angle in CC'. Draw the lines $A'C$, $B'C$, and the circle $O(3k)$. Call S, S' the points of intersection of the lines and the circle on the same side of O as A and B. Draw OS and OS'. These are the alleged trisectors of the given angle.

(b) In the same construction as in (a), draw OD, bisecting the angle AOC. Through C draw a line parallel to OD, meeting the circle $O(3k)$ in S, defined above. Similarly for S'.

No proofs are given. Every construction is incorrect, even when the given angle is a right angle and the true positions of the trisectors are easily found.

The message contained in this pamphlet is not only valueless, but is actually harmful.

VIRGIL SNYDER

NEW BOOKS RECEIVED

Mathematics of Flight. By J. Naidlich. New York and London, McGraw-Hill Book Co., 1943. 10+409 pages. $2.00.

1000 Pre-flight Problems. By W. H. Thompson and M. L. Aiken. New York, Harper and Brothers, 1943. 15+169 pages. $0.88.

Solution of the Trisection Problem. By E. Vennigerholz. Moscow, Idaho. E. Vennigerholz, 1943. 16 pages. $1.25.

The Problem of Moments. By J. A. Shohat and J. D. Tamarkin. (Mathematical Surveys, No. 1.) New York, American Mathematical Society, 1943. 14+140 pages. $2.50.

Gallileo. By C. de Losada y Puga. Lima, Universidad Catolica del Peru. 54 pages.

Meromorphic Functions and Analytic Curves. By H. Weyl and F. J. Weyl. (Annals of Mathematics Studies, No. 13.) Princeton, University Press; London, Humphrey Milford and Oxford University Press, 1943. 9+269 pages. $3.50.

Plane Trigonometry with Tables. By H. D. Ballou and F. H. Steen. Boston, Ginn and Co., 1943. 6+123+84 pages. $2.50.

Elements of Statistical Method. By A. E. Waugh. Second edition. New York and London, McGraw-Hill Book Co., 1943. 21+532 pages. $4.00.

Commercial Algebra. By H. E. Stelson and H. F. Rogers. New York, The Macmillan Co., 1943. 11+283 pages. $2.50.

The Theory of Rings. By N. Jacobson. (Mathematical Surveys, No. 2.) New York. American Mathematical Society, 1943. 6+150 pages. $2.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 591. *Proposed by C. J. Coe, University of Michigan*

Two given coplanar circles, (A_1) and (A_2), are cut orthogonally by a third circle, (B). Prove that a line joining either intersection point on (A_1) to either intersection point on (A_2) will pass through one of two points on the line of centers A_1A_2, these two points being the same for all choices of the orthogonal circle (B).

E 592. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the numbers

$$729,\ 71289,\ 7112889,\ 711128889, \cdots,$$

$$7744,\ 97970404,\ 997997004004,\ 9997999700040004, \cdots$$

are perfect squares. Generalize the latter sequence to an arbitrary scale of notation (with base greater than 4).

E 593. *Proposed by N. A. Court, University of Oklahoma*

A variable tetrahedron has three fixed vertices, a fixed circumsphere, and the sum of the squares of its edges is constant. Find the locus of its Monge point.

E 594. *Proposed by F. J. Duarte, Caracas, Venezuela*

Solve the equations

$$x^2 = y^3, \qquad (x - 3)y + (1 - a - b)(3y - 2x) = 0,$$

where

$$ab = -1, \qquad a^3 + b^3 = 1.$$

E 595. *Proposed by H. T. R. Aude, Colgate University*

Find the smallest set of three different integers to represent the sides of a triangle in which one angle is 60° and each of the other angles differs therefrom by not more than one minute.

SOLUTIONS

The Euler Lines of Three Triangles

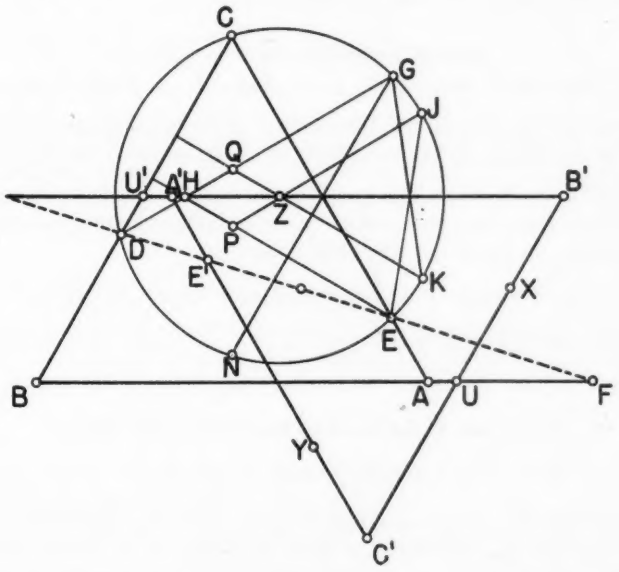
E 547 [1942, 683]. *Proposed by V. Thébault, San Sebastián, Spain*

A diameter d of the circumcircle of an equilateral triangle ABC cuts the sides BC, CA, AB in points D, E, F. Prove that the Euler lines of the three triangles AEF, BFD, CDE form a triangle symmetrically equal to ABC, the center of symmetry lying on d.

Solution by W. B. Clarke, San Jose, California.

LEMMA. *If a triangle CDE has angle $C=60°$ or $120°$, its Euler line forms an equilateral triangle with the lines CD and CE.*

Proof of Lemma. Let Z and H be the circumcenter and orthocenter of the special triangle CDE. Let DH meet the circumcircle again in G, and let the line through G parallel to CD meet the circumcircle again in N. Let J and K be the midpoints of the arcs EC and GN. Then the lines DHG, JZ (perpendicular to CE) and EH, KZ (perpendicular to CD) form a parallelogram $HPZQ$, with P on JZ and Q on KZ.



Since both CG and JK are arcs of 60°, we have $CJ=GK=KN$, whence

$$\angle CEJ = \angle KGN.$$

But $\angle PEC = 30° = \angle QGN$. Hence, by addition or subtraction,

$$\angle PEJ = \angle QGK.$$

Also $\angle EPJ = 60° = \angle GQK$, and $EJ = JC = GK$. Hence the triangles EJP and GKQ are congruent, $JP = KQ$, $ZP = ZQ$, the parallelogram $HPZQ$ is a rhombus, and its diagonal HZ makes an equilateral triangle with CD and CE, as desired.

Proof of Theorem. By the lemma, the Euler lines of AEF, BFD, CDE form a triangle $A'B'C'$ whose sides are the parallels to BC, CA, AB through the respective circumcenters X, Y, Z. Let $B'C'$ meet AB in U, and let BC meet $A'B'$ in U'. Then we can compute

$$A'U' = AF^2/(AB + 3AF) = AU,$$

whence $A'B' = AB$, and the triangles ABC, $A'B'C'$ are not merely similar but congruent. Moreover, if DEF meets $A'C'$ in E', we find $A'E' = AE$. It follows that the center of symmetry is the midpoint of EE'.

Remarks. The triangle XYZ is equilateral, and its circumcircle passes through the center of ABC. More generally, it can be proved that the triangles ABC, $A'B'C'$ are congruent even if they are not equilateral, and even if the transversal d does not pass through the circumcenter of ABC; but in the latter case the center of symmetry will no longer lie on d.

Parabolic Sections of a Degenerate Quadric

E 549 [1942, 683]. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

The face planes of a proper tetrahedron intersect a circumscribed quadric cone in four parabolas. What conditions are thus imposed on the cone?

Solution by the Proposer. The quadric cone under consideration cannot have an accessible vertex, but must reduce to a parabolic cylinder. For, let its equation, referred to the given inscribed tetrahedron, be

$$a_{12}x_1x_2 + a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 + a_{24}x_2x_4 + a_{34}x_3x_4 = 0.$$

The condition for degeneracy is

$$\det |a_{ij}|_4 = 0,$$

where $a_{ji} = a_{ij}$ and $a_{ii} = 0$. If the plane at infinity has the equation

$$a_{01}x_1 + a_{02}x_2 + a_{03}x_3 + a_{04}x_4 = 0,$$

the parabolic sections require the vanishing of the remaining fourth-order principal minors of $\det |a_{ij}|_5$. But it has been proved by V. W. Parker (Bull. Amer. Math. Soc., vol. 38, 1932, pp. 259–262) that if the real symmetric determinant $D = \det |a_{ij}|_5$ with $a_{ii} = 0$ has all five of its fourth-order principal minors zero, then $D = 0$. It follows that our quadric touches the plane at infinity and hence is a parabolic cylinder.

Cylinder and Helicoid

E 551 [1943, 59]. *Proposed by J. H. Butchart, Grinnell College*

Show that the arcs common to a circular cylinder and a right helicoid whose axis lies on the cylinder are pieces of a circular helix.

Solution by C. E. Springer, University of Oklahoma. On a common arc of the right helicoid

$$x = u\cos v, \qquad y = u\sin v, \qquad z = av$$

and the cylinder $x^2+y^2=2rx$, we have $u=2r\cos v$. Hence the common arc has the parametric equations

$$x = 2r\cos^2 v, \qquad y = r\sin 2v, \qquad z = av,$$

which by a change of origin become

$$x = r\cos 2v, \qquad y = r\sin 2v, \qquad z = av.$$

Also solved by Howard Eves, Jerzy Szmojsz, and the proposer.

Squares with Repeated Digits

E 552 [1943, 59]. *Proposed by V. Thébault, San Sebastián, Spain*

In certain scales of notation, a number of the form *aabb* can be the square of a number of the form *cc*, where c is a multiple of b. Show that (1) a and b are relatively prime; (2) b and $c^2/b-a$ are perfect squares.

Solution by Free Jamison, U. S. Navy Air Navigation School. From the multiplication worked out on the right, with radix r, we see that

$$c^2 = qr + b, \qquad r = q + b, \qquad q = a - 1.$$

Thus

$$c^2 = q^2 + (q+1)b = (a-1)^2 + ab.$$

Hence b, which was given to be a divisor of c, is also a divisor of $(a-1)^2$. This makes a and b relatively prime. Suppose that $c=mb$ and $(a-1)^2=nb$; then

$$m^2b = c^2/b = n + a.$$

Therefore n and b are relatively prime; and since their product is a perfect square, each is a perfect square. So also is

$$(a-1)^2/b = (c^2 - ab)/b = c^2/b - a.$$

In the simplest instances, we have $a=r-3$, $b=4$, $c=r-2$. It seems that the smallest values with $c>b>4$ are

$$a = 26391, \qquad b = 16900, \qquad c = 33800, \qquad r = 43290.$$

Also solved by D. H. Browne and E. P. Starke, though the former was not aware that b could be greater than 4.

Cospherical Circles

E 553 [1943, 59]. *Proposed by N. A. Court, University of Oklahoma.*

If two of the four circles of intersection of two spheres with two planes are cospherical, prove that the remaining two circles are likewise cospherical.

Solution by E. P. Starke, Rutgers University. Let $S_1=0$, $S_2=0$, $P_1=0$, $P_2=0$ be the equations of the two spheres and the two planes, and assume that in S_1 and S_2 the coefficient of $x^2+y^2+z^2$ is unity. Then a value of k exists for which $S_1+kP_1=0$ represents any sphere through the circle determined by $S_1=0$ and $P_1=0$. A similar remark applies to $S_2+k'P_2=0$. By hypothesis, there exist values of k and k' such that

$$S_1 + kP_1 \equiv S_2 + k'P_2$$

(since a common sphere passes through the two circles). But this same identity in the form

$$S_1 - k'P_2 \equiv S_2 - kP_1$$

expresses the fact that a certain sphere passes through the other two circles, $S_1=P_2=0$ and $S_2=P_1=0$.

Also solved by C. E. Springer.

The proposer remarks that the two planes may be inverted into two further spheres, with the following result. If it is possible, in one way, to group four given spheres into two pairs so that the circle of intersection of the first pair (real or imaginary) and the circle of intersection of the second pair are cospherical, then the grouping can be done in two other ways. Moreover, the centers of the four spheres are coplanar, and the four spheres belong to the same coaxal net.

Cutting the Cheese

E 554 [1943, 59]. *Proposed by J. L. Woodbridge, Philadelphia*

Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

Solution by Free Jamison, Pittsburgh. Since n straight lines can divide a plane into $(n^2+n+2)/2$ areas, the $(n+1)$st plane can be divided by the first n planes into that number of areas. For each of these areas the $(n+1)$st plane divides a piece of cheese already formed into two, and increases the total number of pieces by $(n^2+n+2)/2$. Since $(n^3+5n+6)/6$ gives the number of pieces for $n=1$ or 2, and since

$$\frac{n^3+5n+6}{6} + \frac{n^2+n+2}{2} \equiv \frac{(n+1)^3+5(n+1)+6}{6},$$

the expression $(n^3+5n+6)/6$ holds for every n.

It is interesting to note that

(1) n points can divide a line into $1+n$ parts,

(2) n lines can divide a plane into $1+n+\binom{n}{2}$ parts,

(3) n planes can divide space into $1+n+\binom{n}{2}+\binom{n}{3}$ parts.

Also solved by D. H. Browne, W. E. Buker, and the proposer.

Editorial Note. The general formula

$$1+n+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{m},$$

for the case of an m-dimensional cheese, was obtained by L. Schläfli on page 39 of his great posthumous work, *Theorie der vielfachen Kontinuität* (Denkschriften der Schweizerischen naturforschenden Gesellschaft, vol. 38, 1901).

The Genus of a Graph

E 555 [1943, 59]. *Proposed by Howard Eves, Syracuse University*

Consider a rectangular parallelepiped of dimensions $a\times b\times c$, made up of abc unit cubes. Imagine the edges of all these unit cubes replaced by material wires, common edges sharing the same wire. Prove that the exterior surface of the resulting wire network is of genus $p=2abc+bc+ca+ab$ (*i.e.*, that it is topologically equivalent to a sphere with p handles).

I. *Solution by C. E. Springer, University of Oklahoma.* Five cuts will render one unit cube topologically equivalent to a sphere. The desired genus is then $5abc$ less the number of common faces of the unit cubes, or

$$p = 5abc - \{(a-1)bc + (b-1)ca + (c-1)ab\} = 2abc + bc + ca + ab.$$

II. *Solution by R. C. Buck, Harvard University.* The wires form a "graph" with V vertices and E edges, where

$$V = (a+1)(b+1)(c+1) = abc + (bc+ca+ab) + (a+b+c) + 1,$$
$$E = a(b+1)(c+1) + b(c+1)(a+1) + c(a+1)(b+1)$$
$$= 3abc + 2(bc+ca+ab) + (a+b+c).$$

If we cut each of the E edges, the resulting array is homeomorphic to V spheres. Joining $V-1$ of these to the remaining one, we obtain a single sphere. Hence the genus of the original surface is

$$p = E - (V-1) = 2abc + (bc+ca+ab).$$

Editorial Note. The principle involved in the latter solution may be expressed as follows. (Cf. Kurt Reidemeister, *Einführung in die kombinatorische Topologie*, Braunschweig, 1932, p. 106, or Dénes König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936, pp. 53–57.) A finite, connected graph, with V nodes and E branches, has for its "skeleton" or "scaffolding" (Gerüst) a tree with the same nodes and $V-1$ of the branches. The connectivity (Zusammenhangszahl) of the graph is defined as the number of its branches that have to be removed to make the tree, namely $E-V+1$. This is the same as the genus of the surface obtained by "blowing up" the graph (*i.e.*, replacing its edges by material wires); for the blown-up tree is homeomorphic to a sphere, and each extra branch can be regarded as a handle. The notion of the connectivity of a graph occurred simultaneously to Kirchhoff (*Annalen der Physik und Chemie*, vol. 72, pp. 498, 506) and von Staudt (*Geometrie der Lage*, Nürnberg, 1847, pp. 20–21).

The Turning Point of a Quadratic Function

E 556 [1943, 119]. *Proposed by C. W. Bruce, Wesleyan College, Macon, Georgia*

The graph of a quadratic function passes through three given points (x_i, y_i). Show that the abscissa of its maximum or minimum is

$$\begin{vmatrix} x_1^2 & y_1 & 1 \\ x_2^2 & y_2 & 1 \\ x_3^2 & y_3 & 1 \end{vmatrix} \div 2 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Solution by Virginia Carlton, Wesleyan College, Macon, Georgia. Since the curve

$$ax^2 + bx + c = y \tag{1}$$

passes through the three given points, we have

$$ax_i^2 + bx_i + c = y_i \qquad (i = 1, 2, 3). \tag{2}$$

Eliminating a, b, c from the four equations (1) and (2), we obtain the same curve in the alternative form

$$\begin{vmatrix} x^2 & x & 1 & y \\ x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{vmatrix} = 0.$$

When we differentiate, the first row becomes

$$2x \quad 1 \quad 0 \quad dy/dx.$$

Setting the derivative equal to zero, and expanding, we obtain

$$2x\begin{vmatrix} x_1 & 1 & y_1 \\ x_2 & 1 & y_2 \\ x_3 & 1 & y_3 \end{vmatrix} - \begin{vmatrix} x_1^2 & 1 & y_1 \\ x_2^2 & 1 & y_2 \\ x_3^2 & 1 & y_3 \end{vmatrix} = 0,$$

and it remains to transpose the second and third columns of each determinant.

Also solved by R. K. Allen, Howard Eves, P. G. Smith, E. P. Starke, Eugene Titus, C. W. Topp, Alan Wayne, Hazel S. Wilson, and the proposer.

A Sphere Rolling on a Fixed Sphere

E 557 [1943, 120]. *Proposed by V. Thébault, San Sebastián, Spain*

A sphere (S) of constant radius rolls on a fixed sphere (O) in such a way as to pass through a fixed point A. Determine the loci of the centers of similitude of the spheres (S) and (O).

Solution by Howard Eves, Syracuse University. Let M be the internal, N the external, center of similitude, and let S and O be the centers of the spheres. Since length OM, length ON, line OA, and angle AOM are all fixed, it is evident that the loci of M and N are circles having OA as an axis. The locus of M is also, of course, on the sphere (O).

Also solved by the proposer.

Areas and Distances

E 558 [1943, 120]. *Proposed by V. V. Nákladem, Philadelphia*

Let P be any point in the plane of a triangle ABC. Show that the sum of the squares of the areas of the three triangles PBC, PCA, PAB cannot exceed

$$(PA^2 + PB^2 + PC^2)^2/16.$$

I. *Solution by J. B. Kelly, Brown University.* Let $a, b, c, \alpha, \beta, \gamma$ denote the lengths PA, PB, PC and angles BPC, CPA, APB. Then the sum of the squares of the three areas is given by

$$S = (b^2c^2 \sin^2 \alpha + c^2a^2 \sin^2 \beta + a^2b^2 \sin^2 \gamma)/4.$$

We maximize S, regarding a, b, c fixed, and α, β, γ variable, with the relation

$$\alpha + \beta + \gamma = 2\pi \tag{1}$$

connecting them. Applying the method of Lagrange, we have

$$\sin 2\alpha = \lambda/b^2c^2, \qquad \sin 2\beta = \lambda/c^2a^2, \qquad \sin 2\gamma = \lambda/a^2b^2. \tag{2}$$

Then, since $2 \sin^2 x = 1 - \sqrt{1 - \sin^2 2x}$, the maximum value sought is given by

$$8S = b^2c^2 + c^2a^2 + a^2b^2 - \sqrt{b^4c^4 - \lambda^2} - \sqrt{c^4a^4 - \lambda^2} - \sqrt{a^4b^4 - \lambda^2}. \tag{3}$$

From (1) we have

$$\sin 2\alpha = -\sin (2\beta + 2\gamma) = -\sin 2\beta \cos 2\gamma - \sin 2\gamma \cos 2\beta,$$

or, by virtue of (2),

$$\frac{\lambda}{b^2c^2} = -\frac{\lambda}{c^2a^2}\sqrt{1-\frac{\lambda^2}{a^4b^4}} - \frac{\lambda}{a^2b^2}\sqrt{1-\frac{\lambda^2}{c^4a^4}},$$

which reduces to

$$2\sqrt{b^4c^4-\lambda^2} = a^4 - b^4 - c^4.$$

Substitution of this and two similar expressions in (3) yields

$$16S = (a^2+b^2+c^2)^2.$$

It is evident that this value of S is a maximum. Thus the theorem is established.

II. *Solution by the Proposer.* Let P, A, B, C have Cartesian coordinates $(0, 0)$, (x_i, y_i) $(i=1, 2, 3)$, respectively. Using a different interpretation for the same coordinates, consider vectors $X=(x_1, x_2, x_3)$ and $Y=(y_1, y_2, y_3)$. Then we have

$$\begin{aligned} 4(PBC^2+PCA^2+PAB^2) &= \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}^2 + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}^2 + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 \\ &= |X\times Y|^2 \leqq X^2Y^2 \\ &\leqq (X^2+Y^2)^2/4 = (PA^2+PB^2+PC^2)^2/4. \end{aligned}$$

A Question from Plato

E 560 [1943, 120]. *Proposed by S. H. Gould, University of Toronto*

In the fifth book of his *Laws*, the philosopher Plato, discussing the distribution of land in a colony, seeks a number divisible by every integer from 1 through 10 and chooses 5040. Show in general that if m and n are positive integers with $n<p$, where p is the smallest prime greater than m, then $m!$ is divisible by n except when $m=3$.

Solution by Arthur Rosenthal, University of New Mexico. If $n\leqq m$ there is no question; so let us suppose that $m<n<p$. Then n is composite, say $n=n_1n_2$ (with $2\leqq n_1\leqq n_2$).

Suppose first that $n_1<n_2$. By Tschebyscheff's Theorem, there is at least one prime between m and $2m$. Hence $n<p<2m$ and

$$n_1 < n_2 \leqq n/2 < m.$$

Thus $m!$ is divisible by n_1 and n_2, and hence by n also.

If $m>3$, a refinement of Tschebyscheff's Theorem gives $m<p<2m-2$, whence

$$n \leqq 2m-4 = m(2-4/m) \leqq m\cdot m/4 = (m/2)^2.$$

Thus in the remaining case $n_1=n_2\leqq m/2$. Hence $m!$ is divisible by $n_1\cdot 2n_1$, and so also by $n_1^2=n$. Among the cases $m=1, 2, 3$, only $m=3$ gives an exception.

Also solved by D. H. Browne, Walter Penney, E. P. Starke, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4097. *Proposed by R. Bellman, Madison, Wisconsin*

Let

$$f(x) = 1 + \sum_{n=1}^{\infty} a_{1n}x^n, \qquad [f(x)]^k = 1 + \sum_{n=1}^{\infty} a_{kn}x^n.$$

What is the necessary and sufficient condition that the following system of equations have a solution

$$x_0 + \sum_{n=1}^{N} a_{kn}x_n = u_k; \qquad k = 1, 2, \cdots, N+1; N \geqq 1.$$

4098. *Proposed by P. R. Halmos, Urbana, Illinois*

Evaluate the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ x_1^4 & x_2^4 & x_3^4 & \cdots & x_n^4 \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ x_1^{2^{n-1}} & x_2^{2^{n-1}} & x_3^{2^{n-1}} & \cdots & x_n^{2^{n-1}} \end{vmatrix}.$$

4099. *Proposed by J. H. Butchart, Grinnell College*

If P is any point of a curve and Q is the corresponding point of the pedal with respect to the point O, then OQ makes the same angle with the pedal that OP makes with the curve.

4100. *Proposed by V. Thébault, San Sebastián, Spain*

Let $N = 1, 2, 3, \cdots, n$ be a number in the system of base $n+1$ where N is written with the consecutive increasing digits omitting 0. The product $P = N \cdot L$ is formed where $L = \alpha\beta$ is a number of two digits such that $\gamma = \alpha + \beta$ is a number less than n and such that γ and n have δ as the greatest common divisor. Show that

$$P = N \cdot L = ab \cdots pqab \cdots pq \cdots ab \cdots p(n-\gamma)q,$$

the $\delta-1$ periods $ab, \cdots, pq$ being formed by n/δ distinct digits and the number of $(n/\delta)+1$ digits on the right contains the digit $(n-\gamma)$ between the digits p and q.

Dedicated to E. P. Starke.

SOLUTIONS

Differential Equation, Curvature

4037 [1942, 340]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Integrate

$$(x^{n+1}+y^n)y'-x^ny=0;$$

calculate and examine the radius of curvature of the integral curves at the origin.

Solution by Paul D. Thomas, U. S. Coast and Geodetic Survey, Sherburne, N. Y. Writing the given differential equation thus

$$\left(\frac{x}{y}\right)^n\frac{ydx-xdy}{y^2}-\frac{dy}{y^2}=0,$$

the integral is seen to be

$$(1)\qquad \frac{1}{n+1}\left(\frac{x}{y}\right)^{n+1}+\frac{1}{y}=C, \quad \text{or} \quad x^{n+1}-C(n+1)y^{n+1}+(n+1)y^n=0.$$

From (1) it is seen that for $n\geqq 2$, the curves have a cusp at the origin, and the radius of curvature at this point is therefore 0. For $n=1$, the integral curves are the conics $x^2-2Cy^2+2y=0$, the radius of curvature at the origin being $+1$. For the computation write the given differential equation in the form $M\,dx+N\,dy=0$. Then the radius of curvature will be given by

$$(2)\qquad R=\frac{(M^2+N^2)^{3/2}}{MN(M_y+N_x)-N^2M_x-M^2N_y},$$

$$M=x^ny, \quad M_x=nyx^{n-1}, \quad M_y=x^n, \quad N=-x^{n+1}-y^n,$$

$$N_x=-(n+1)x^n, \quad N_y=-ny^{n-1}.$$

Let $x=r\cos A$, $y=r\sin A$. With $C=0$, (1) may be written $r=-(n+1)\tan^nA\sec A$. With these and the above values of M, N, M_x, M_y, N_x, N_y (2) becomes

$$R=-\frac{\tan^{n-1}A}{n}\cdot[(n+1)^2\cdot\tan^2A+n^2]^{3/2}.$$

From this last with $A=0$, $n=1$, $R=-1$. For $n\geqq 2$, $R=0$.

Editorial Note. The computations may be made directly from the equation in polar coordinates, and it will be seen that for the approximate formulas needed here for the origin neighborhood the term with the arbitrary constant coefficient may be neglected. Thus for $n\geqq 2$ we have

$$r \to -(n+1)\theta^n, \quad dr/d\theta \to -n(n+1)\theta^{n-1}, \quad d^2r/d\theta^2 \to -n(n^2-1)\theta^{n-2},$$
$$R \to -n^2\theta^{n-1}.$$

Thus the integral curve is tangent to the x-axis at the origin; and, if n is also an even positive integer, there is a cusp at the origin, the two branches lying respectively in the second and third quadrants, whereas for n odd the curve passes from the third to the fourth quadrant and tangent to the x-axis at the origin.

Powers of a Point

4040 [1942, 408]. *Proposed by N. A. Court, University of Oklahoma*

Given n points by $A_1, A_2, \cdots, A_n$ in space, let G denote their centroid, k^2 the sum of the squares of the $n(n-1)/2$ segments determined by the given points, and P_{ij} the power of a given point P with respect to the sphere having for diameter the segment A_iA_j; show that

$$\sum P_{ij} = n(n-1)PG^2/2 - k^2/2n.$$

Solution by Paul D. Thomas, U. S. Coast & Geodetic Survey, Little Falls, N. Y. Consider the points A_i, A_j the midpoint R of A_iA_j, and the point P. If P_{ij} is the power of P with respect to the sphere upon A_iA_j as diameter,

$$2P_{ij} = 2\overline{PR}^2 - \overline{A_iA_j}^2/2. \tag{1}$$

Since PR is a median of triangle PA_iA_j, $2\overline{PR}^2 = \overline{PA_i}^2 + \overline{PA_j}^2 - \overline{A_iA_j}^2/2$, and substituting in (1) gives $2P_{ij} = \overline{PA_i}^2 + \overline{PA_j}^2 - \overline{A_iA_j}^2$.

Thus it is easily seen (or readily proved by mathematical induction) that

$$2\sum P_{ij} = (n-1)\sum_{i=1}^{n} \overline{PA_i}^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{A_iA_j}^2_{i\neq j},$$

or

$$2\sum P_{ij} = (n-1)\sum_{i=1}^{n} \overline{PA_i}^2 - k^2. \tag{2}$$

But

$$\sum_{i=1}^{n} \overline{PA_i}^2 = n\overline{PG}^2 + k^2/n, \tag{3}$$

See Court's article in the *National Mathematics Magazine*, 15, 1941, p. 273, art. 4. Substituting (3) in (2) gives the announced relation.

Solved also by Howard Eves, C. E. Springer, and the proposer.

Editorial Note. The solutions of Eves and Springer derived the relation (3), and Eves remarked that the theorem of the problem and its proof applies to any euclidean space.

First Order Ordinary Differential Equations

4041 [1942, 408]. *Proposed by Cezar Coşniţă, Focşani, Roumania*

Integrate the following differential equations

$$y' = \frac{x^2 - y^2 + 1}{x^2 - y^2 - 1}, \qquad y' = -\frac{y(2x + y - 1)}{x(x + 2y - 1)}.$$

Determine the integral curves passing through the origin.

Solution by B. Z. Linfield, University of Virginia. Writing the equations in the form $M\,dx+N\,dy=0$ (or sums of such forms), they become

$$(x^2 - y^2)(dx - dy) + dx + dy = 0,$$
$$(x + y - 1)(y\,dx + x\,dy) + xy(dx + dy) = 0.*$$

The first has $(x+y)^{-1}$ for an obvious integrating factor, yielding

$$\frac{(x - y)^2}{2} + \log\frac{x + y}{C} = 0, \quad \text{or} \quad (x + y)e^{(x-y)^2/2} = C.$$

The second is exact, yielding

$$xy(x + y - 1) = C.$$

The integral curves of the first form therefore a continuous family of probability curves having the line $x+y=0$ for their common asymptote when $C\neq 0$, together with this asymptote when $C=0$, and thus cover the entire plane. In other words, for any (finite) point in the plane this equation has one, and only one, integral curve passing through it. And, according as this point *is* or *is not* on the line $x+y=0$, the integral curve through it is this line or a probability curve. For this equation, in the form $M\,dx+N\,dy=0$, there is no (finite) point that makes its M and N both zero.

On the other hand, for the second equation, in the same form, there are four "singular points" that make its M and N both zero, namely the vertices of the triangle

$$xy(x + y - 1) = 0, \quad \text{and its centroid} \quad (1/3, 1/3).$$

For any point in the plane, except these four "singular points," there is one, and only one, integral curve passing through it. If the point is not on a "side" (line) of the last triangle, the corresponding integral curve is a cubic of the family $xy(x+y-1)=C$, having the "sides" (lines) of this triangle for asymptotes, and having its maxima and minima on a median of this triangle.

According as this point is *inside* or *outside* the triangle, the part of the integral curve through it *is* or *is not* an oval. In the 1st case $0>27C>-1$. In the 2nd

* The arrangement of these sums is clearly determined by symmetry (elegance) and exactness. All the integral curves here are beautiful to look at, but are left for the reader to draw.

case, either the point lies in an "alternate" angle, and $27C<-1$; or it doesn't, and $C>0$. If the point is on a "side" (line) of the triangle (but not one of the vertices), the corresponding integral curve is that line. However, corresponding to a vertex of the triangle, there are two integral curves through it, namely the lines of the triangle through it.

Corresponding to the point (1/3, 1/3), the integral curve

$$27xy(x+y-1) = -1$$

is a solution of the differential equation, and this point satisfies its equation. But (1/3, 1/3) is an isolated point of this curve. (The point and the rest of the curve are separated by the triangle as in the case of the oval.) Hence, in the strict sense of the terms, there is *no* integral curve through the point (1/3, 1/3). Still, all integral curves not through the "singular points" are asymptotic to those through them, and are non-degenerate and non-singular.

It would seem, therefore, that for neither of these two differential equations has the origin any properties that other points (finite or infinite in number) do not have. The point (1/3, 1/3) is, however, "singular" (exceptional) for the second equation, and is different from the remaining three "singular points," the vertices of the triangle $xy(x+y-1)=0$.

The *moral* in these differential equations seems to be, therefore, to watch the "singular points" that make M and N both zero in $M\,dx+N\,dy=0$, or that makes $F(x, y)$ "indeterminate" in $y'=F(x, y)$; and the integral curves through them.† Having never seen a Roumanian calculus, and having no desire to cause any foreign entanglements, the writer cannot help speculating that had all authors of the calculus called explicit attention to the last *moral*, without fail, this problem might not have been proposed at all. Also, students of the calculus would then be better prepared to continue with mathematics.

This solution of the problem is clearly related to the work of Henri Poincaré as discussed by Marston Morse (*What is Analysis in the Large?*) in the same number of the MONTHLY where this problem appeared.

Solved also by G. A. Baker, Cecil Hastings, Jr., C. E. Springer, and P. D. Thomas.

Minimum of a Determinant

4042 [1942, 408]. *Proposed by Henry Scheffé, Princeton University*

Prove that if A is a fixed positive definite hermitian matrix and X is a variable non-negative hermitian matrix (rank = index), then the minimum value of the determinant $|A+X|$ is $|A|$ and is attained if and only if $X=0$.

Solution by the Proposer. There exists a square matrix P such that

$$P^*AP = I. \tag{1}$$

Then

† And hence the points that make the integrating factor infinite or zero.

$$P^*(A + X)P = I + P^*XP. \tag{2}$$

Since P^*XP is non-negative hermitian, there exists a unitary matrix S_X (depending on X),

$$S_X^* S_X = I, \tag{3}$$

such that

$$S_X^*(P^*XP)S_X = (y_i\delta_{ij}),$$

a diagonal matrix with y_i real and $\geqq 0$. Applying S_X^* on the left and S_X on the right of (2) yields

$$S_X^* P^*(A + X)PS_X = I + (y_i\delta_{ij}). \tag{4}$$

Taking determinants in (1), (3), (4), we get

$$|S_X^*| \cdot |S_X| \cdot |P^*| \cdot |P| \cdot |A + X| = \prod_i (1 + y_i),$$

$$|P^*| \cdot |P| \cdot |A| = 1, \qquad |S_X^*| \cdot |S_X| = 1,$$

whence

$$|A + X| = |A| \prod_i (1 + y_i).$$

The minimum is obviously $|A|$ and is obtained if and only if all $y_i = 0$, that is if and only if $X = 0$. The theorem was stated originally for real matrices but was altered for hermitian matrices at the suggestion of Orrin Frink, Jr.

Pedal Spheres of a Tetrahedron

4044 [1942, 408]. *Proposed by V. Thébault, San Sebastián, Spain*

Determine the straight lines such that the circumsphere of the pedal tetrahedron of each of its points with respect to any given tetrahedron $ABCD$ passes through a fixed point P.

Examine the case for which $ABCD$ is orthocentric and P is the foot of one of its altitudes.

Editorial Note. The proposer remarked that this problem was suggested by the following theorem, a verification of which he had requested of Cl. Servais:

The pedal spheres of points P of a given straight line relative to the tetrahedron $ABCD$ cut a certain sphere orthogonally.

Servais made this the starting point for important studies published in the *Bulletins de l'Académie royale de Belgique*, 1922, 52. This theorem is true when the sphere reduces to a point. The planes perpendicular to AP, BP, CP, DP at P meet the faces BCD, CDA, DAB, ABC in the straight lines ($\alpha\alpha_1$, $\beta\beta_1$, $\gamma\gamma_1$, $\delta\delta_1$) which are generators of the same system of a ruled surface for which four directrices (d_1, d_2, d_3, d_4) are also on the surface Ω of the 4th order, the locus of

the points for which the pedal spheres with respect to $ABCD$ pass through P. The desired lines are the above eight straight lines and only these.

The case where $ABCD$ is orthocentric and P is one of the feet of its altitudes is very simple and curious. This case was treated by the proposer before proposing to Servais the more general problem.

Euler's Constant

4045 [1942, 479]. *Proposed by A. M. Glicksman, The Bronx High School of Science*

Show that γ, Euler's constant, is given by

$$\gamma = \sum_{r=2}^{\infty} (-1)^r \frac{g_r}{r}, \qquad g_r = \sum_{k=1}^{\infty} \frac{1}{k^r}, \qquad \gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right).$$

Solution by the Proposer. We have

$$\log\left(1+\frac{1}{k}\right) = \frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \cdots \qquad (k \geqq 1).$$

The sum of a finite number of convergent series is convergent, and therefore

$$\sum_{k=1}^{n-1} \log\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n-1} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n-1} \frac{1}{k^2} + \frac{1}{3}\sum_{k=1}^{n-1} \frac{1}{k^3} - \cdots.$$

But:

$$\sum_{k=1}^{n-1} \log\left(1+\frac{1}{k}\right) = \log \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3} \cdots \frac{n}{n-1} = \log n.$$

Hence:

$$\frac{1}{2}\sum_{k=1}^{n-1} \frac{1}{k^2} - \frac{1}{3}\sum_{k=1}^{n-1} \frac{1}{k^3} + \cdots = \sum_{k=1}^{n-1} \frac{1}{k} - \log n = -\frac{1}{n} + \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

(At this point, we might say let $n\to\infty$ and the theorem is proved. However, this requires justification, which is accomplished as follows):
Now the series:

$$s(n) \equiv \left(\frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{k^2}\right) - \left(\frac{1}{3}\sum_{k=n}^{\infty} \frac{1}{k^3}\right) + \left(\frac{1}{4}\sum_{k=n}^{\infty} \frac{1}{k^4}\right) - \cdots$$

is a series of alternately positive and negative terms which diminish towards zero. Hence $s(n)$ converges, and is less in value than its first term. From which it follows that $\lim_{n\to\infty} s(n) = 0$. Add $s(n)$ to both sides of the last equation above, obtaining:

$$\frac{1}{2}g_2 - \frac{1}{3}g_3 + \frac{1}{4}g_4 - \cdots = s(n) - \frac{1}{n} + \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right).$$

We may now let $n\to\infty$, thus completing the proof.

Orthocentric Tetrahedron and Euler Lines

4049 [1942, 480]. *Proposed by V. Thébault, San Sebastián, Spain*

In an orthocentric tetrahedron $ABCD$ the straight lines joining the centroid with the orthocenter of the triangles of the faces cut the respective radical planes of the circumsphere and the spheres with the medians of $ABCD$ as diameters in four points of the same plane.

Solution by N. A. Court, University of Oklahoma. I. Consider the general tetrahedron $(T)=ABCD$, and let (Q) be its "quasi-polar" sphere, *i.e.*, the sphere having for center the Monge point M of (T) and for the square of its radius one third of the power of M for the circumsphere (O) of (T).* This sphere is orthogonal to the four spheres having the medians of (T) for diameters.†

THEOREM A. *The radical plane of the circumsphere of a tetrahedron (T) and the sphere having for diameter a median of (T) is the polar plane of the centroid of the respective face of (T) with respect to the quasi-polar sphere of (T).*

Let (AG_a) be the sphere having for diameter the median AG_a of (T). This sphere being orthogonal to (Q), the points A, G_a are conjugate with respect to (Q), hence the polar plane λ of the centroid G_a of the face BCD passes through A and is perpendicular to the line MG_a, say, at the point K, which point lies thus on the sphere (AG_a). Now the orthogonal projection of the mid-point of AG_a upon the plane λ is the mid-point of the segment AK, hence AK is a diameter of the circle along which the plane λ cuts the sphere (AG_a).

If O, G are the circumcenter and the centroid of (T), we have, both in magnitude and in sign, $MG=GO$, $AG=3GG_a$. Applying Menelaus' theorem to the triangle AGO and the transversal MG_a, we find that the point of intersection of MG_a with AO is the diametric opposite of A on the circumsphere (O) of (T).

Hence (O) passes through the point K, and AK is a diameter of the circle along which the sphere (O) is cut by the plane λ. Thus the plane λ cuts the two spheres (AG_a), (O) along the same circle and is therefore their radical plane.

THEOREM B. *The four radical planes of the circumsphere of a tetrahedron (T) and the spheres having the medians of (T) for diameters meet the corresponding faces of the polar reciprocal tetrahedron of (T) with respect to the quasi-polar sphere of (T) in four lines lying in the same plane.*

The radical plane λ of the circumsphere (O) and the sphere (AG_a) being the polar plane of G_a for (Q), the plane λ meets the polar plane α of A for (Q) in the polar line, say, m of AG_a for (Q). Now the median AG_a contains the centroid G of (T), hence the polar plane ϕ of G for the sphere (Q) passes through the line m.

Similarly for the other vertices of (T). Hence the proposition.

* N. A. Court, On the theory of the tetrahedron, Bulletin of the American Mathematical Society, Vol. 48, 1942, p. 583.

† *Ibid.*, p. 587. Also this MONTHLY, Vol. 39, 1932, pp. 196, 197.

II. If the tetrahedron (T) becomes an orthocentric tetrahedron (T_h), its Monge point coincides with its orthocenter H, and the quasi-polar sphere becomes the polar sphere (H) (see, for instance, the writer's *Modern Pure Solid Geometry*, p. 265, art. 813). Moreover, (T_h) coincides with its polar reciprocal with respect to the sphere (H). The two theorems proved above thus become:

THEOREM A′. *The radical plane of the circumsphere of an orthocentric tetrahedron (T_h) and a sphere having for diameter a median of (T_h) is the polar plane of the centroid of the respective face of (T_h) with respect to the polar sphere of (T_h).*

THEOREM B′. *The four radical planes of the circumsphere of an orthocentric tetrahedron (T_h) with the spheres having for diameters the medians of (T_h) meet the respective faces of (T_h) in four lines lying in the same plane.*

The plane of the four lines is the polar plane of the centroid G of (T_h) with respect to the sphere (H), *i.e.*, the orthic plane of (T_h).

The four points in which the radical planes considered meet the Euler lines of the respective faces of (T_h) obviously lie in the orthic plane of (T_h) which proves the proposed proposition. Moreover, this proposition would remain valid if instead of the Euler lines four arbitrary lines were taken in the four faces of (T_h).

The propositions A' and B' may be proved directly, in the same manner as the propositions A and B were proved.

Note. The corresponding proposition for the triangle is due to V. Thébault (this MONTHLY, vol. 49, 1942, p. 63, question E 467). It may proved be in a similar manner.

Editorial Note. All of the above theorems may be extended to the non-degenerate simplex S in n dimensions with the $n+1$ vertices A_i, the centroid G, and the circumcenter C. The *Monge planes* π_{ij} are defined as the hyperplanes perpendicular to the edges A_iA_j and passing respectively through the centroid G_{ij} of the remaining $n-1$ vertices. With an arbitrarily chosen origin of vectors the equation of π_{ij} is

$$(1) \qquad \pi_{ij}\colon\ (\mathbf{a}_i - \mathbf{a}_j)\cdot[\mathbf{x} - (n+1)\mathbf{g}/(n-1)] + (\mathbf{a}_i^2 - \mathbf{a}_j^2)/(n-1) = 0,$$

where $\mathbf{g}$ is the vector of G and $\mathbf{a}_i$ that of A_i. This system of $n(n+1)/2$ equations has one and only one solution for the vector $\mathbf{x}$ since the system of the same number equations

$$(\mathbf{a}_i - \mathbf{a}_j)\cdot\mathbf{x} - (\mathbf{a}_i^2 - \mathbf{a}_j^2)/2 = 0$$

has one and only one solution which is the vector $\mathbf{c}$ of C. Hence, if we denote by $\mathbf{m}$ the vector solution of (1), we must have

$$(2) \qquad (n+1)\mathbf{g} = 2\mathbf{c} + (n-1)\mathbf{m},$$

and the point whose vector is $\mathbf{m}$ is called the *Monge point* M.

We now take the origin of vectors at M, and it is easily seen that

$$(3) \qquad (n+1)\mathbf{g}\cdot\mathbf{a}_i - \mathbf{a}_i^2 = nm, \qquad 2\sum \mathbf{a}_i\cdot\mathbf{a}_j = n(n+1)m, \qquad 2\mathbf{c} = (n+1)\mathbf{g}.$$

The scalar m may be positive, negative, or zero, but it will be convenient at present to exclude the value zero. Let (M) denote the sphere with center M and radius $\sqrt{m}$, and $[P]$ the polar of G with respect to (M). We easily obtain the following equations

$$(4) \qquad \begin{aligned} &(M)\colon \mathbf{x}^2 - m = 0; \qquad (C')\colon n\mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + m = 0; \\ &(G)\colon \mathbf{x}^2 - 2\mathbf{g}\cdot\mathbf{x} + m = 0; \qquad (C)\colon \mathbf{x}^2 - 2\mathbf{c}\cdot\mathbf{x} + nm = 0; \\ &[P]\colon \mathbf{g}\cdot\mathbf{x} - m = 0; \end{aligned}$$

where (C') is the inverse of (C) with respect to (M), (G) has center G and is orthogonal to (M), and $[P]$ is the common radical plane of the four spheres. These are the same equations as in (4) in the solution of 3963 [1942, 133] with H replaced by M.

If we denote by (A_iG_i) the sphere with diameter A_iG_i where G_i is the centroid of the remaining n vertices, we have for its equation

$$(5) \qquad \begin{aligned} (A_iG_i)\colon \mathbf{x}^2 - [2\mathbf{c} + (n-1)\mathbf{a}_i]\cdot\mathbf{x}/n + m = 0, \\ 4n^2r_i^2 = (n+1)^2(\mathbf{g} - \mathbf{a}_i)^2; \end{aligned}$$

where r_i is its radius. The radical plane of (A_iG_i) and (C) has the equation $\mathbf{g}_i\cdot\mathbf{x} - m = 0$, $n\mathbf{g}_i = 2\mathbf{c} - \mathbf{a}_i$, where $\mathbf{g}_i$ is the vector of G_i. We easily obtain the identity

$$(6) \qquad n\{G_i\} + \{A_i\} \equiv (n+1)\{G\};$$

where $\{A_i\} \equiv \mathbf{a}_i\cdot\mathbf{x} - m$ and $\{A_i\} = 0$ is the equation of the polar of A_i with respect to (M), *etc.* Thus any configuration of points common to the polars of A_i and G_i belongs also to the polar of G. For $n = 2$ the point M is in all cases the orthocenter. A necessary and sufficient condition that S be orthocentric with the origin at M and $n > 2$ is that $\mathbf{a}_i\cdot(\mathbf{a}_j - \mathbf{a}_k) = 0$, i, j, k distinct; then it follows that $\mathbf{a}_i\cdot\mathbf{a}_j$ is a constant and this constant must be m. In this case S is self-polar with respect to (M).

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Louise Adams has been appointed assistant professor at High Point College, North Carolina.

Assistant Professor C. K. Alexander of Occidental College has been appointed associate professor and head of the department of mathematics.

Assistant Professor Harriet W. Allen of Hollins College has been promoted to an associate professorship.

Dr. H. G. Ayre of Western Illinois State Teachers College has been promoted to an associate professorship.

Assistant Professor H. M. Bacon of Stanford University has been promoted to an associate professorship.

Dr. G. A. Baker of the College of Agriculture of the University of California has been promoted to an assistant professorship.

Assistant Professor Juna L. Beal has been appointed acting head of the department of mathematics at Butler University.

Associate Professor A. C. Berry of Lawrence College, Wisconsin, has been promoted to a professorship.

Dr. B. H. Bissinger and Dr. Fritz Herzog of Cornell University have been appointed instructor and assistant professor, respectively, at Michigan State College.

Assistant Professor J. W. Blincoe of the University of Tennessee has been appointed to a professorship at Randolph-Macon College.

Dr. J. O. Blumberg of the University of Pittsburgh has been promoted to an assistant professorship.

W. H. Bradford of the John McNeese Junior College of the Louisiana State University has been promoted to an assistant professorship and appointed head of the department of mathematics and science.

Assistant Professor A. D. Bradley of Hunter College became a lieutenant (j.g.) in the U.S.N.R. and is now teaching in the Naval Air Station in Dallas, Texas.

R. L. Calvert of the Utah State Agricultural College has been promoted to an assistant professorship.

Dr. Max Dehn of Illinois Institute of Technology has been appointed professor of mathematics at St. John's College, Annapolis.

Professor Ralph Hull of the University of British Columbia has been appointed professor and chairman of the department of mathematics at the University of Nebraska.

Mrs. Phyllis H. Hutchings has been appointed assistant professor of astronomy and mathematics at Whitman College.

Associate Professor Nathan Jacobson of the University of North Carolina has been appointed associate professor at Johns Hopkins University.

Dr. S. A. Jennings of the University of British Columbia has been appointed to an assistant professorship.

Dr. G. W. Mackey of Illinois Institute of Technology has been appointed to an instructorship at Harvard University.

Dr. Haim Reingold, Supervisor of Mathematics Instruction in the Signal Corps Training Schools of the Illinois Institute of Technology, has been appointed assistant professor of mathematics in that institution.

Assistant Professors M. A. Sadowsky and L. R. Wilcox of Illinois Institute of Technology have been promoted to associate professorships.

Dr. C. L. Seebeck, Jr., of the University of Alabama has been promoted to an assistant professorship.

Dr. W. S. Snyder of Illinois Institute of Technology has been appointed assistant professor at the University of Tennessee.

Assistant Professor T. L. Wade, Jr., of the University of Alabama has been appointed head of the department of mathematics at the Florida State College for Women.

The following appointments to instructorships are announced:
Illinois Institute of Technology: Furio Alberti, Dr. Ruth Mason Ballard, P. L. Browne, H. J. Miser
Massachusetts Institute of Technology: G. N. Raney
Whitman College: Mrs. Pearl C. Miller
University of Rochester: H. P. Atkins

Professor Emeritus E. B. Van Vleck of the University of Wisconsin died on June 2, 1943.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

ESSENTIAL MATHEMATICS FOR MINIMUM ARMY NEEDS

In this department of the MONTHLY, August–September, 1943, attention was called to a report on Pre-Induction Courses in Mathematics issued by a committee of the United States Office of Education. Since the writing of that report, the War Department, believing that a supplementary statement of a more specific nature would be helpful in carrying out the philosophy outlined in that basic report, requested the United States Office of Education to cooper-

ate in further study of the subject. The members of the committee receiving the new assignment were Virgil S. Mallory, State Teachers College, Montclair, New Jersey, Chairman; Rolland R. Smith, Public Schools, Springfield, Massachusetts; C. Louis Thiele, Public Schools, Detroit, Michigan; F. L. Wren, George Peabody College for Teachers, Nashville, Tennessee; William A. Brownell, Duke University; and Wells Harrington, representing the Civilian Pre-Induction Training Branch of the War Department. John Lund and Giles M. Ruch represented the U. S. Office of Education.

The complete report of the new Committee appeared in the October issue of the *Mathematics Teacher*. Also a brief statement of the findings appeared in *Education for Victory*, September 1, 1943. In view of the importance of the project, Professor Virgil S. Mallory was invited to write a digest of the work of the Committee. The following paragraphs are due to him.

The procedure employed by the Committee was to confer with Army officers directly in charge of the training of enlisted men, and to observe the basic training process itself during the first thirteen weeks of the inductee's Army life. To systematize both interviews and observations, a check list of one hundred forty-one items was employed. This check list, carefully prepared by the Committee, had the benefit of criticism from more than twenty Army training officers stationed in the Washington area. It was then tried out in the original and in a revised form with a dozen training officers in nearby camps. In its final form its contents provided the basis for informal conferences with another group of ninety-six training officers and for item-by-item checking by one hundred seventy-eight officers who were serving as instructors in basic training in seventy-four different kinds of Army jobs in Replacement Training Centers and unit training centers in eight states. The centers visited are operated by branches which together train seventy-five per cent of the enlisted men of the Army.

The purpose of the Committee's investigation was to determine those items in mathematics which should make up the *minimum* equipment of the inductee. Every inductee might well have more than this minimum, but he cannot have less and successfully meet the demands of basic training. It should be emphasized that many enlisted men go on for specialized training which calls for more mathematics than is outlined in the report. Such men can profit by as much of the sequential work in mathematics as they can master.

The list of essential topics in mathematics for minimum Army needs is not presented by the Committee as a course of study. The responsibility for determining the nature and details of such a course rests with local school authorities. There is no thought that only the items listed should be taught, and that these should be taught only within the limits suggested. Nor is there any thought of asking capable students to substitute a course based on the listed items for the sequential mathematics of the senior high school. For those not taking such courses, instruction in the essential mathematics listed in the report should be carried as far as time permits. Moreover, there are other mathematical concepts

and skills, not found to be part of the necessary minimum equipment, but almost certainly valuable in the Army, on which instruction might well be offered.

It is a fallacy to assume that enrollment in advanced high school courses in mathematics assures proficiency in the minimum essentials listed in the report. Ample evidence in the Army and in civilian life shows that the study of algebra and geometry does not guarantee the maintenance of lower-order mathematical abilities and knowledge. On this account, school officials and teachers should take steps to ascertain whether students taking advanced mathematics possess the needed minimum essentials, and then to teach whatever may be necessary. It should be noted in this connection that the earlier report already referred to suggested changes in the sequence of courses to give time for much of the material of the present report.

The casual reader of the report can easily oversimplify the problem if he thinks of the essentials listed purely in terms of mechanical, paper-and-pencil computation. Proficiency in computation is of course necessary, but it should not be mere mechanical proficiency. Much more is called for than computational competence.

On this last point the testimony of training officers is unequivocal. They say that many enlisted men, even those who seem to be able to obtain correct answers in abstract computation, are unable to think quantitatively; that is to say, they cannot use in practical situations even the limited skills which they possess. The implications of this charge (and it was universal) are unmistakable, and they cannot be disregarded. What is needed is a reorientation, a change of emphasis, in instruction. Computation has too often been stressed, and accurate, skilled thinking in concrete quantitative situations has been minimized. Many students have acquired tricks with numbers which have proved valueless under conditions of use. Meanwhile two aspects of mathematical learning have suffered, namely, understanding and experience in application.

It follows that refresher courses in the high school and elsewhere are not the sole remedy for the present emergency. Such courses may only restore former skills which are inadequate for Army needs and for the demands of civilian life. Instead, young men about to enter the Army must be taught something which heretofore has not been taught often enough, namely, the ability to meet quantitative problems effectively and confidently. They must be able to identify the quantitative aspects of the situations which confront them, to deal with these situations by approximation and estimation when computation is not required, to recognize and use the simpler symbolism of mathematics, to tell when and how mathematical symbolism, concepts, and processes are to be employed, and to compute accurately, quickly, and intelligently when computation is called for. Courses with objectives less ambitious than these are of limited value.

Some additional explanation is desirable in regard to the check list which was used by members of the Committee in their visits to Army camps. The items on the list were concerned with mathematical skills, concepts, and relationships as well as with memorized formulas and rules; they covered the field of arithmetic,

and dealt with the simpler aspects of algebra, geometry, and trigonometry. With many of the items an example was given to set a working level of difficulty. Only the uses of mathematics in the thirteen weeks of basic training given to all inductees were to be recorded.

As already indicated, a total of one hundred seventy-eight instructors actually engaged in teaching inductees filled out the check list. They were asked to do four things: (1) to mark each item as of frequent, occasional, or rare use; (2) to classify the examples given as of less, the same, or greater difficulty than inductees would actually need; (3) to furnish practical examples of how the mathematical items listed are used; and (4) to report on mathematical needs not covered in the list. Each instructor was advised before marking the check list to be concerned only with the mathematical needs in his own particular field of instruction, not with his philosophy about needs in general.

In selecting from the items on the check list those which should be recommended for inclusion in the outline of essential mathematics for Army needs, attention was given first of all to frequency of use. On this basis a number of the items obviously had to be rejected. But it soon became apparent that frequency of use alone was not a sufficient basis for rejecting or including an item. In many cases, an item which was indicated as rarely used might still be of fundamental importance. The conclusions of the Committee appear in the following outline.

Outline of Essential Mathematics for Minimum Army Needs

A. *Reading and Writing Arithmetical Symbols.* Whole numbers (to six places). Common fractions, especially those with denominators which are powers of 2 through 64, and with denominators 3, 5, 6, 10, and 12; fractions with denominators in the ten-thousands are frequently used in map reading. Decimal fractions (to three places); in machine work it is sometimes necessary to read micrometer calipers to ten-thousandths. Per cents (to 100%); in certain instances per cents less than 1% are used to indicate the composition of metals and the proportions of chemicals in solutions.

B. *Counting.* Counting by 1's, 2's, 5's, and 10's (to 500).

C. *Operations with Whole Numbers.* Addition (columns of not more than five addends of three digits each, and shorter columns of not more than six digits). Subtraction (numbers of not more than six digits). Multiplication (numbers of four digits by numbers of two digits). Division (numbers of four digits divided by numbers of two digits).

D. *Operations with Common Fractions.* The denominators are limited as in the second topic of (A).

E. *Operations with Decimal Fractions.* The fractions are restricted as in the third topic of (A).

F. *Part-Whole Relationships*, with comon fractions, decimal fractions, and per cents. Finding part of a quantity. Finding what part one number is of another. Finding a number, given a part and its relative size.

G. *Ratio and Proportion.* A study of the basic idea with problem solving.

H. *Powers and Roots.* Finding powers (squares and cubes). Finding squares and square roots from tables.

I. *Graphs and Maps.* Understanding grids and scales. Determining directions from a map. Interpreting and making maps and graphs.

J. *Tables.* Reading tables.

K. *Formulas and Equations.* Understanding of simple symbolism of algebra. Simple formulas to be memorized: area of circle, triangle, rectangle; volume of cylinder, rectangular solid; distance, rate, and time relationships. Substitution in simple formulas. Solution of simple equations.

L. *Positive and Negative Numbers.* Symbolism and meaning.

M. *Measurement,* including understanding of basic units. Length, weight, area, and volume. Temperature (C and F), angles (degrees, minutes, and seconds), time (24-hour clock). Metric system with simple equivalents. Measuring instruments. Limits of accuracy or tolerances. Estimation.

N. *Geometric Concepts.* Point; straight, curved, horizontal, vertical, oblique, parallel, and perpendicular lines; angle; and slope. Triangle (right, scalene, isosceles, and equilateral); parallelogram (square and rectangle); trapezoid; circle; ellipse; regular polygon; prism; cylinder; cone; and sphere. Similarity.

O. *Drawing and Construction.* Use of ruler graduated in 32ds and in 10ths of an inch, and in millimeters; use of compasses to construct circles; use of protractors to measure and to draw angles. Construction of a perpendicular to a line. Understanding of views of a simple object as given in a scale drawing, blueprint, or sketch. Knowledge of the 3-4-5 relation of the sides of a right triangle.

P. *Miscellaneous.* Averages: mean, median, and mode. Rounding off numbers.

The report then gives, in considerable detail, suggestions for teaching each of the items included in the above list. More space is devoted to topics related to the teaching of arithmetic than to those topics with which high school teachers are most familiar. Throughout this part of the report, emphasis is placed on the following points.

(1) *Meaning and understanding.* "Learning should proceed from the concrete (where meaning is most easily apprehended) to the abstract (where the idea or skill is freed of its particularized content) and back again to the concrete (where the freed idea or skill can be applied usefully)."

(2) *Applications.* From the standpoint of learning, providing ample experiences in application completes the cycle from concrete to abstract and back again to concrete.

(3) *Estimation and approximation.* It is not always necessary and is frequently impracticable to use pencil and paper to find answers in Army situations. In such cases the soldier must estimate and approximate, and he will be successful in so doing only to the extent (a) that he understands meanings, (b) that he has what may be called "mathematical sense," and (c) that he de-

velops an appreciation of relative size both with abstract number and with units of measure.

The following paragraphs typify the summary to the report.

"The items listed previously must be part of the equipment of all Army inductees. On this point there is no distinction among prospective inductees who are in private schools or in public schools, who have left school for no particular reason, or who have positions in industry. In the age population concerned there is no distinction between prospective inductees who are in high school and those who are in the grades, or, within the high school, between those who are taking the four-year mathematics sequence, and those who have had one year or less of mathematics since the eighth grade. This fact carries a number of implications for persons charged with the responsibility for organizing instructional programs.

"Students majoring in mathematics need to be examined carefully to make sure that they do have the needed ideas and skills. To the extent that they do not, provision should be made in each advanced course to correct all deficiencies. (And it should be repeated that this provision is inadequate if it consists only in "refresher" practice, to re-instate useless number tricks.) Furthermore, the content of these advanced courses should be studied carefully with a view (a) to eliminating material which will not function and (b) to including vital applications arising from the emergency. In this connection, mathematics teachers should consult the report of the Office of Education Mathematics Committee, in which many valuable suggestions are offered. Also, they might well make use of the Seventeenth Yearbook of the National Council of Teachers of Mathematics (1942), entitled "A Source Book of Mathematical Applications."

"As for a special course, to provide students in school with the mathematical essentials, school officials must think in terms of the *ages* of students, not in terms of the *grade* level they may have reached. Physically able 16- and 17-year-old prospective inductees will soon be in military service, whether at present they are high school students or are enrolled in the eighth grade. Steps should be taken to bring together in convenient groupings all students who are about to be inducted or are about to leave school for work. It is obvious that the schools offer the readiest means for assembling prospective inductees for instructional purposes.

"Prospective inductees already out of school, for whatever reason, should be given an opportunity to make up their mathematical deficiencies. It is often possible to have special classes in public school buildings late in the afternoon or in the evening. Many business colleges are already offering such courses as are many private industrial concerns. These agencies of pre-induction training should be encouraged and multiplied.

"Let the mathematics course be taught by persons who know mathematics, not alone in the technical sense, but in the sense of concrete, sensible, practical application. Such teachers will be able to tell when their students have acquired the ideas and skills set in the minimum list."

A NEW DEFERMENT PROCEDURE FOR MATHEMATICIANS

In April, 1943, a deferment procedure for mathematicians, involving the National Committee on Physicists and Mathematicians, was authorized under Activity and Occupation Bulletin No. 35. This was explained fully in a memorandum distributed to chairmen of departments of mathematics on May 11, 1943, and reproduced in the June-July issue of this MONTHLY. By this procedure employers were advised to send the original copy of Form 42A to the National Committee for evaluation. From the Committee, the form was returned to the local board with an appropriate statement by the Committee concerning the advisability of the deferment. The National Committee was also empowered to appeal cases of registrants for whom occupational deferment was not granted by the local board. Under this arrangement, the Committee has performed its duties well and effectively.

Activity and Occupation Bulletin No. 35, which authorizes the activities of the National Committee, is to be rescinded in the near future, over the strong protests of representatives of mathematics and physics.* The Committee will therefore cease to exist and will no longer be available to advise local boards on problems connected with the deferment of mathematicians. The new procedure for persons engaged in the occupations defined as critical in Local Board Memorandum No. 115, revised August 16, 1943, is described in Local Board Memorandum 115B. *Only those cases in which occupational deferment is refused by the local board will become involved in this procedure.*

In the August 16 revision of Local Board Memorandum 115, a list of critical occupations, which includes mathematicians, was set up.

The following represents a summary of the occupational deferment procedure for the critical occupations:

1. The employer continues, as in the past, to present his case for deferment to the local board through the use of Form 42A.
2. The local board will arrive at a classification for the registrant on the basis of the evidence submitted to it by the employer. It may, if it so desires, consult with the local United States Employment Service office concerning classification. (LBM 115 and LBM 115-C)
3. The local board will then make its decision. If occupational deferment is granted, no further action is involved during the period covered by the deferment. If, however, deferment is refused, and the registrant is placed in I-A, the case is continued as follows.
4. *If no appeal is taken* within the 10-day period allowed for this purpose, the local board is *directed* to refer the case to the local (local with respect to the registrant's local board) office of the United States Employment Service and a 30-day stay of induction is granted to permit action by this office.

* The essential part of Bulletin No. 35 was rescinded October 21.—Ed.

5. *If an appeal is taken* and the local board's decision is reversed, the registrant will be reclassified, and the case is closed for the duration of the deferment period. If, however, the I-A classification made by the local board is supported by the appeal board, the case *must be* referred to the local U.S.E.S. office.
6. When the local U.S.E.S. office receives a case from the local board either directly (4, above), or after appeal (5, above), it may certify to the local board either that

   (a) the registrant should be deferred in his present position, or
   (b) that they have succeeded in placing the registrant in a new position.

   In the first case the local board is directed to reopen the classification of the registrant, and presumably will grant occupational deferment. In the second case, a further period of 10 days is given the registrant for the filing of a new Form 42A requesting occupational deferment in his new position, and on the basis of which the local board shall reopen the classification.
7. If the U.S.E.S. fails to present any notification to the local board during the period allowed, or if it certifies that the registrant is needed in new employment, but does not succeed in placing the man in such employment, the local board may proceed with the induction of the registrant.
8. The local offices of the United States Employment Service can refer cases reaching them and involving persons in "critical" professional and scientific occupations to the National Roster of Scientific and Specialized Personnel. The details of the relation between the U.S.E.S. offices and the Roster have not been fully defined.

The following suggestions are made to facilitate the proper functioning of the procedure herein outlined.

(1) It becomes of even greater importance than heretofore that an employer's original presentation of a case to the local board for occupational deferment or continuation of deferment be made as strong as possible. No opportunity for strengthening a case should be overlooked.

(2) The employer having once requested occupational deferment should be prepared to appeal every case in which such deferment is refused by the local board. This again implies that the original presentation should be as strong as possible, so as to stand up under an appeal.

(3) If an appeal does not secure deferment, the employer should take it upon himself to see that the case is referred to the local U.S.E.S. office, as directed by Local Board Memorandum 115-B. It is possible that the local board may not be familiar with the requirements that they do this.

(4) The employer should take steps to insure that a case which has reached the U.S.E.S. office is actually referred by it to the National Roster.

(5) It should be noted that there is no clear statement as to whether an appeal may be taken *after* the action of the U.S.E.S. Presumably an appeal could be taken at this time, but it is implied in LBM 115-B that the proper time for an appeal is immediately (within 10 days) after the local board's original I-A classification.

(6) In case the procedures here listed have been followed and misclassification of important personnel nevertheless results, the National Roster should be informed regarding the particulars of the case.

The Secretary of the War Policy Committee will be interested in learning of the experiences of department chairmen with the new procedure. Such information will give our representatives in Washington and the War Policy Committee a basis for judging the effectiveness of the plan.

October 20, 1943.

J. R. KLINE,
Secretary, War Policy Committee

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-SIXTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-sixth summer meeting of the Mathematical Association of America was held at the New Jersey College for Women, Rutgers University, New Brunswick, New Jersey, on Saturday and Sunday, September 11–12, 1943, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meeting of the Institute of Mathematical Statistics. Three hundred and fourteen persons were in attendance at the meetings, including the following one hundred seventy-three members of the Association:

C. R. ADAMS, Brown University
LOUISE ADAMS, High Point College
E. B. ALLEN, Rensselaer Polytechnic Institute
C. B. ALLENDOERFER, Haverford College
R. D. ANDERSON, University of Texas
H. E. ARNOLD, Wesleyan University
L. A. AROIAN, Hunter College
W. L. AYRES, Purdue University

A. V. BAEZ, Wagner College
FRANCES E. BAKER, Vassar College
D. H. BALLOU, Middlebury College
J. A. BENNER, Lafayette College
GARRETT BIRKHOFF, Harvard University
G. D. BIRKHOFF, Harvard University
B. H. BISSINGER, Cornell University
E. E. BLANCHE, Curtiss-Wright Corporation
T. A. BOTTS, U. S. N. R.
JULIA W. BOWER, Connecticut College
J. G. BOWKER, Middlebury College
H. W. BRINKMANN, Swarthmore College
B. H. BROWN, Dartmouth College
C. T. BUMER, Kenyon College
L. H. BUNYAN, Rutgers University
HERBERT BUSEMANN, Illinois Institute of Technology

HOBART BUSHEY, Hunter College
JEWELL H. BUSHEY, Hunter College

S. S. CAIRNS, Queens College
MILDRED E. CARLEN, Brown University
W. B. CARVER, Cornell University
JOHN CAWLEY, Lafayette College
F. L. CELAURO, Syracuse University
J. O. CHELLEVOLD, Wartburg College
MARY D. CLEMENT, Wells College
A. B. COBLE, University of Illinois
H. R. COOLEY, New York University
T. F. COPE, Queens College
RICHARD COURANT, New York University
C. C. CRAIG, University of Michigan
E. L. CROW, Bureau of Ordnance, Navy Department
FORREST CUMMING, University of Georgia
H. B. CURRY, Frankford Arsenal
J. H. CURTISS, Bureau of Ships, Navy Department
E. H. CUTLER, Lehigh University

D. R. DAVIS, State Teachers College, Montclair, N. J.
F. F. DECKER, Syracuse University
R. H. DOWNING, Kaiser Cargo, Inc.
ARNOLD DRESDEN, Swarthmore College
W. H. DURFEE, Yale University
O. L. DUSTHEIMER, John Carroll University

BENJAMIN EPSTEIN, Frankford Arsenal
J. P. EVERETT, Western Michigan College
G. M. EWING, University of Missouri

W. H. FAGERSTROM, College of the City of New York
A. B. FARNELL, U. S. Military Academy
F. A. FICKEN, University of Tennessee
W. W. FLEXNER, Cornell University
TOMLINSON FORT, Lehigh University
R. M. FOSTER, Bell Telephone Laboratories
J. S. FRAME, Michigan State College
ORRIN FRINK, JR., Pennsylvania State College

M. G. GALBRAITH, Rutgers University
R. E. GASKELL, Brown University
B. P. GILL, College of the City of New York
H. S. GRANT, Rutgers University
J. W. GREEN, Aberdeen Proving Ground
J. A. GREENWOOD, Bureau of Aeronautics, Navy Department
J. I. GRIFFIN, College of the City of New York
C. C. GROVE, College of the City of New York
V. G. GROVE, Michigan State College

V. H. HAAG, Hershey Junior College
THEODORE HAILPERIN, Cornell University
D. W. HALL, University of Maryland
P. R. HALMOS, Syracuse University
G. E. HAY, University of Michigan
G. A. HEDLUND, University of Virginia
M. H. HEINS, Illinois Institute of Technology
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
T. R. HOLLCROFT, Wells College
L. C. HUTCHINSON, Brown University
EMMA HYDE, Kansas State College

B. W. JONES, Cornell University

AIDA KALISH, Brooklyn, N. Y.
EDWARD KASNER, Columbia University
J. R. KLINE, University of Pennsylvania
HELEN L. KUTMAN, Hunter College

W. D. LAMBERT, U. S. Coast and Geodetic Survey
V. V. LATSHAW, Lehigh University
H. L. LEE, University of Tennessee
SOLOMON LEFSCHETZ, Princeton University
JOSEPH LEHNER, Kellex Corporation
MARGUERITE LEHR, Bryn Mawr College
D. C. LEWIS, JR., University of New Hampshire
MARIE LITZINGER, Mount Holyoke College
R. T. LUGINBUHL, University of Pennsylvania

C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MAC LANE, Harvard University
H. F. MACNEISH, Brooklyn College
MARGARET P. MARTIN, DeLamar Institute of Public Health
W. T. MARTIN, Syracuse University
REV. P. H. MCGRATH, St. Peter's College
E. J. MCSHANE, University of Virginia
A. E. MEDER, JR., New Jersey College for Women
L. L. MERRILL, Rensselaer Polytechnic Institute
F. H. MILLER, Cooper Union
E. B. MODE, Boston University
DEANE MONTGOMERY, Princeton University
R. K. MORLEY, Worcester Polytechnic Institute
RICHARD MORRIS, Rutgers University
D. S. MORSE, Union College
E. J. MOULTON, Northwestern University

REV. PAUL MUEHLMANN, West Baden College
F. D. MURNAGHAN, Johns Hopkins University
W. R. MURRAY, Franklin and Marshall College

C. A. NELSON, New Jersey College for Women
P. B. NORMAN, New York University

C. O. OAKLEY, Haverford College
F. W. OWENS, Pennsylvania State College

GORDON PALL, McGill University
CHARLES PFLAUM, University of Pennsylvania
A. E. PITCHER, Lehigh University
W. G. POLLARD, University of Tennessee
HILLEL PORITSKY, General Electric Company

J. F. RANDOLPH, Cornell University
C. H. RAWLINS, JR., U. S. Naval Academy
G. E. RAYNOR, Lehigh University
O. H. RECHARD, University of Wyoming
C. J. REES, University of Delaware
MINA S. REES, Hunter College
C. F. REHBERG, New York University
R. G. D. RICHARDSON, Brown University
J. F. RITT, Columbia University
R. E. ROOT, U. S. Naval Academy
JOSEPH ROSENBAUM, Bloomfield, Conn.
J. B. ROSSER, Cornell University
S. G. ROTH, Washington Square College

J. M. SACHS, U. S. N. R.
RAPHAEL SALEM, Massachusetts Institute of Technology
HENRY SCHEFFÉ, Princeton University
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K. W. WEGNER, Carleton College
ANNA PELL WHEELER, Bryn Mawr College
P. M. WHITMAN, University of Pennsylvania
G. T. WHYBURN, University of Virginia
S. S. WILKS, Princeton University

R. C. YATES, U. S. Military Academy
BERTRAM YOOD, U. S. N. R.

OSCAR ZARISKI, Johns Hopkins University

Convenient dormitory rooms were available for mathematicians and their families, and excellent meals were served in Cooper Hall. On Saturday afternoon a very pleasant reception for those attending the meetings was given by Dean Margaret T. Corwin of the New Jersey College for Women at the Dean's residence; and on Sunday evening members of the Department of Music of the College presented an excellent musical program.

The joint dinner for the three organizations was held on Sunday evening at 6:30. Professor Morris acted as toastmaster, and he first introduced Dean Corwin who extended a very gracious welcome to the visitors. Dean Richardson of Brown University was the next speaker. He called attention to the relative

neglect of the field of applied mathematics in America and to the present increase of interest in this direction and the special work which Brown University is now doing in this field. Professor G. D. Birkhoff spoke on the development of interest in higher mathematics in the universities of Mexico and South America. On motion of Professor Ayres, resolutions were adopted by a rising vote extending thanks to President Clothier of Rutgers University, Dean Corwin of the College for Women, Professor Morris, chairman of the Committee on Arrangements, and Mr. Snedecker, in charge of business management of the college, for the many facilities of the college which had been placed at the disposal of the convening organizations, and for the attention to the details of the arrangements that contributed so largely to the enjoyment of the meetings.

The American Mathematical Society held sessions on Sunday and Monday. As the twenty-fifth Colloquium, Professor E. J. McShane of the University of Virginia delivered three lectures on the subject "Existence theorems in the calculus of variations." On Sunday afternoon there were two lectures by invitation, the first by Professor Antoni Zygmund of Mount Holyoke College on "The complex method of the theory of trigonometric series," and the second by Professor F. D. Murnaghan of Johns Hopkins University on "Finite deformations of an elastic solid."

The Institute of Mathematical Statistics held sessions on Sunday morning and afternoon and a joint session with the Society on Monday morning.

The Mathematical Association held sessions on Saturday morning, afternoon and evening. Credit for the very interesting program should go to the program committee, which consisted of Professors G. A. Hedlund, chairman, C. C. MacDuffee, and G. B. Price. The program follows.

First Session of the Association

1. "Introduction of a Riemannian geometry on a manifold" by Professor S. S. Cairns, Queens College.
2. "Mathematical models" by Professor Frances Baker, Vassar College.
3. "Mathematical properties of military maps" by Professor W. W. Flexner, Cornell University.

Second Session of the Association

1. "Mathematics in Mexico and South America" by Professor G. D. Birkhoff, Harvard University.
2. "Curves and Surfaces" by Professor J. W. T. Youngs, Purdue University.
3. "Some neglected work of James Stirling and Karl Schellbach on interpolation" by Professor I. J. Schoenberg, University of Pennsylvania.

Third Session of the Association

1. "The Navy V-12 college training program" by Dean I. C. Crawford, School of Engineering, University of Michigan.

2. "The Navy V-12 training program at Dartmouth" by Professor B. H. BROWN, Dartmouth College.

3. "The Army B and C premeteorology training programs at Brown University" by Professor J. D. TAMARKIN, Brown University.

4. "The application of mathematics in the Army A training program in meteorology" by Professor B. HAURWITZ, Massachusetts Institute of Technology.

MEETING OF THE BOARD OF GOVERNORS

Nine members of the Board, including two regional governors, were present at the meeting held Sunday morning.

The following sixty persons were elected to membership on applications duly certified:

FURIO ALBERTI, B.S.(Chicago) Instr., Illinois Inst. of Tech., Chicago, Ill.

A. H. ANDERSON, M.E.(Marquette) Head of Science Dept., Whitefish Bay Schools, Whitefish Bay, Wis.

C. S. BANWARTH, M.S.(Notre Dame) Instr., Signal Corps Training, Illinois Inst. of Tech., Chicago, Ill.

P. R. BARTRAM, A.B.(Hamilton) Clerk, Bethlehem Steel Co., Buffalo, N. Y.

E. M. BEESLEY, Ph.D.(Brown) Asst. Prof., Univ. of Nevada, Reno, Nevada

GARRETT BIRKHOFF, A.B.(Harvard) Asso. Prof., Harvard Univ., Cambridge, Mass.

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H. D. BRUNK, A.M.(Rice) Fellow, Rice Institute, Houston, Texas

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PAUL CIVIN, Ph.D.(Duke) Instr., Univ. of Buffalo, Buffalo, N. Y.

PAUL D'ARCO, B.S.(Chicago) Instr., Radio Mechanics, Army Air Force Tech. Training Command, Illinois Inst. of Tech., Chicago, Ill.

M. L. DEMOSS, M.S.(Kansas St. T. C., Pittsburg) Instr., General Motors Inst., Flint, Mich.

FLORA DINKINES, A.M.(Michigan) Instr., Univ. of South Carolina, Columbia, S. C.

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R. L. DURHAM, B.S.(Trinity Coll.) President, Southern Sem. and Jr. Coll., Buena Vista, Va.

J. P. ESPOSITO, Ed.M. (DePaul) Teacher, Crane Tech. High School, Chicago, Ill.

A. B. FARNELL, M.S.(Louisiana State) 1st Lt., Instr., U. S. Military Acad., West Point, N. Y.

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MICHAEL GOLOMB, Ph.D.(Berlin) Instr., Purdue Univ., Lafayette, Ind.

THEODORE HAILPERIN, Ph.D.(Cornell) Instr., Cornell Univ., Ithaca, N. Y.

REV. B. A. HAUSMANN, Ph.D.(Yale) Chm. of Dept., Univ. of Detroit, Detroit, Mich.

FRANK HAWTHORNE, B.S. in Educ.(Penn. St. T. C., Edinboro) Instr., Alliance Coll., Cambridge Springs, Pa.

EDWARD HELLY, Ph.D.(Vienna) Visiting Lecturer, Illinois Inst. of Tech., Chicago, Ill.

E. J. HILLS, Ph.D.(Southern California) Instr., Los Angeles City Coll., Los Angeles, Calif.

H. J. JORDAN. M/Sgt., Weather Forecaster, Army Air Forces, Base Weather Office, Smyrna, Tenn.

G. K. KALISCH, Ph.D.(Chicago) Instr., Univ. of Kansas, Lawrence, Kans.

O. J. KARST, A.B.(Montclair State T. C.) Instr., Newark Coll. of Engineering, Newark, N. J.

F. J. KERR. Pleasantville, Pa.

R. A. KLIPHARDT, B.S.(Armour Inst. of Tech.) Grad. Asst., Illinois Inst. of Tech., Chicago, Ill.

J. C. KOKEN, A.B.(Missouri) Instr., Parks Air Coll., East St. Louis, Ill.

P. J. KOPP, A.M.(Duke) Major, Chem. Warfare Service, Washington, D. C.

H. D. LIPSICH, A.B.(Cincinnati) Teaching Fellow, Grad. School, Univ. of Cincinnati, Cincinnati, Ohio

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A. D. SOLLINS. Asst. Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.

PETER THULLEN, Ph.D.(Münster, Germany) Prof., Matematicas Superiores, Escuela de Artilleria e Ingenieros; Director, Departmento Matematico-Actuarial, Instituto Nacional de Prevision, Ecuador

Mrs. CAROL C. VALUCKAS, A.M.(Cornell) Asst., Univ. of New Mexico, Albuquerque, N. M.

ISRAEL WALLACH, M.S.(New York Univ.) Teacher, Thomas Jefferson High School, New York, N. Y.

E. E. WALLICK, Ed.M.(Temple Univ.) Teacher, Junior-Senior High School, Lakewood, N. J.

F. M. WOOD, M.A.(Queen's) Asso. Prof., Civil Engg., McGill Univ., Montreal, P. Q., Canada

A report of the War Policy Committee was presented by Secretary Dresden. An important feature of this report was the fact that the Rockefeller Foundation had made a grant to cover the expenses of the committee. It was also reported that the Committee on Available Teachers of Collegiate Mathematics had accomplished some worthwhile results in the placing of teachers in positions where they were needed.

The Board discussed the desirability of holding mathematical meetings during the period of the war emergency. It was voted to hold the summer and annual meetings for 1944 in conjunction with the American Mathematical Society at times and places to be determined later. A resolution was passed expressing

the opinion that mathematical meetings should be continued during the emergency at least to the extent of two national meetings and the usual smaller meetings, and should be increased in number if the need should become apparent.

The secretary reported that, in line with an earlier action of the Board authorizing the disposal of the Association library, he plans to discontinue most of the arrangements for exchanging the AMERICAN MATHEMATICAL MONTHLY with other periodicals after the end of the year 1943. The Board approved this procedure.

W. B. CARVER, *Secretary-Treasurer*

JOINT MEETING OF THE ILLINOIS, MICHIGAN, AND INDIANA SECTIONS

The first joint meeting of the Illinois, Michigan, and Indiana Sections of the Mathematical Association of America was held at the University of Notre Dame on April 9–10, 1943. This meeting replaced the regular twenty-fourth annual meeting of the Illinois Section, and the twentieth annual meeting of the Indiana Section. The Michigan Section held its regular spring meeting in March.

Forty-six registered at the meeting, including the following thirty members of the Association: H. M. Ackley, Emil Artin, W. L. Ayres, S. F. Bibb, I. W. Burr, C. H. Butler, C. J. Coe, A. H. Copeland, P. D. Edwards, L. R. Ford, J. W. Givens, Jr., M. R. Hestenes, L. S. Johnston, M. W. Keller, E. C. Kiefer, W. C. Krathwohl, E. D. McCarthy, Karl Menger, G. T. Miller, Paul Muehlman, Ivan Niven, P. M. Pepper, G. W. Petrie, E. W. Ploenges, J. C. Polley, W. T. Reid, R. M. Thrall, W. R. Utz, J. W. Wiley, A. J. Zanolar.

At the business meeting of the Indiana Section on Saturday the following officers were elected for the next year: Chairman, P. M. Pepper, University of Notre Dame; Vice-Chairman, Emil Artin, Indiana University; Secretary, M. W. Keller, Purdue University. The next meeting of the Indiana Section will be held at Indianapolis on October 29–30, 1943, in conjunction with the fall meeting of the Indiana Academy of Science.

The meetings on Friday afternoon and evening and on Saturday morning were devoted to a series of lectures arranged by the department of mathematics of the University of Notre Dame. At these sessions the following papers were presented:

1. *The teaching of the calculus of probabilities*, by Professor A. H. Copeland, University of Michigan.

The speaker discussed certain pedagogical problems which arise in teaching the calculus of probability. A brief outline of the subject was given from which the major portion of a course in probability could be obtained by filling in the details of the indicated proofs. In such a course Stieltjes integration is almost

unavoidable, and most students are unfamiliar with this topic. But it was pointed out that statistical considerations lead naturally and simply to an understanding of this integral. The significance of the relation between probabilities and statistical data was also discussed. The speaker presented an outline by means of which the student could acquire an insight into this relation. The presentation of the subject as outlined was recommended as a simplification of much of the material usually presented in courses in probability.

2. *Mathematics and meteorology*, by Professor W. T. Reid, University of Chicago.

This paper was devoted to a description of the '*A*', '*B*', and '*C*' programs of the Army Air Force for meteorological training. Primary attention was given to the position of mathematics in the curricula of these programs.

3. *Mathematics in navigation*, by Ensign A. L. Whiteman, U. S. N., introduced by Professor Karl Menger.

In this paper Ensign Whiteman discussed the place of mathematics in a course in navigation. He pointed out some of the more important problems that arise in navigation, and indicated what mathematics is needed to solve them.

4. *Nomography*, by Professor L. R. Ford, Illinois Institute of Technology.

The content of Professor Ford's address concerned the development of methods by means of which one could teach nomography with the greatest possible simplicity, and at the same time cover the general theory. It was remarked that the material to be taught falls into three parts, namely: (1) the use of three-rowed determinants; (2) the parametric representation of curves; (3) the transformation theory.

Various examples were used to show how to set up the basic determinant from which a chart is made. The speaker explained that the projective transformation arises naturally as a result of multiplying the basic determinant by a determinant with constant elements. He then solved what he called "the problem of the rectangular page," which is the problem of projecting a desired portion of an alignment chart so as to fill up a given page.

5. *On the theory of complex functions*, by Professor Emil Artin, Indiana University.

The simplification of some of the proofs given in courses on complex variables was effected by the use of certain topological notions. The topics treated were Cauchy's integral theorem, Cauchy's formula, singular points, zeros, and analytic continuation.

6. *On calculus*, by Professor Karl Menger, University of Notre Dame.

Professor Menger emphasized the possibility of dividing the statements which occur in the calculus into three groups. Those of the first group do not require the concept of a limit. This group includes in particular the graphical and numerical methods. Those of the second group use the concept of a limit,

but in a rather mild way. This group includes the formal rules of the calculus of derivatives and the calculus of antiderivatives which can be deduced from two rules and two specific derivatives by means of an "Algebra of Analysis." The third group is based upon the existence of a maximum for each continuous function on a closed interval. This group includes the theorems of Rolle, Taylor, etc., and the theory of maxima and minima.

7. *The teaching of two unrelated topics in freshman algebra*, by Professor P. M Pepper, University of Notre Dame.

The first topic discussed was a schematic diagram for solving mixture problems. The second topic was an innovation in the teaching of logarithms. Professor Pepper proposed that the student be introduced to the theory of logarithms by the computation of a two place table of logarithms of the integers from one to ten by means of the equalities (some of which are approximate) $2^{10}=10^3$, $3^9=2\cdot 10^4$, $4=2^2$, $5=10/2$, $6=2\cdot 3$, $7^4=2^3\cdot 3\cdot 10^2$, $8=2^3$, $9=3^2$.

8. *Averages*, by Professor I. W. Burr, Purdue University.

About twenty averages were listed in this paper. Their applications were discussed with special emphasis upon the following fundamental concepts: (1) an average is that constant which for some purpose can replace the individual members; (2) an average may be found by transforming the variable, finding the average of the results, and applying the inverse transformation. The paper was illustrated by applications in various fields.

E. C. KIEFER, *Secretary of the Illinois Section*
C. J. COE, *Secretary of the Michigan Section*
M. W. KELLER, *Secretary of the Indiana Section*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, Chicago, Illinois, November 27–28, 1943.

The following is a list of the Sections of the Associations with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, N. Y., April 22, 1944
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 11, 1944
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CONTENTS

What is Measure? . . . . . S. M. Ulam 597
An Elementary Proof of the Budan-Fourier Theorem . . . . . N. B. Conkwright 603
A Plane Representation of Vectors and Tensors . . . F. P. Beer 605
A Projective Construction for Plane Nodal Cubics . . I. C. Fischer 611
Discussions and Notes . . . . . H. E. Robbins, J. E. Brock, Irving Reiner, H. Schwerdtfeger 617
Clubs and Allied Activities . . . . . J. S. Frame 622
Recent Publications 627
Problems and Solutions 633
News and Notices 643
War Information 645
The Navy College Training Program 645
The Mathematical Association of America 650
Nebraska Section Meeting 650
Wisconsin Section Meeting 652
Metropolitan New York Section Meeting 653
Meetings of the Association and its Sections 655
Index to Volume 50, 1943 656

DECEMBER 1943